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NUMBER
BY DEVELOPMENT

A METHOD
OF NUMBER INSTRUCTION

VOL. III

Grammar
Grades

BY
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SUPERINTENDENT OF SCHOOLS, CHICOPEE, MASSACHUSETTS

PHILADELPHIA & LONDON
J. B. LIPPINCOTT COMPANY

49196

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Printed by J. B. Lippincott Company
The Washington Square Press, Philadelphia, U. S. A.

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INTRODUCTION

The Author's purpose in this volume is to offer to the teacher outlines in detail of the teaching steps in decimal fractions, percentage, interest and its applications, and average, so that she may be able to conduct her work steadily and consistently on a logically arranged development plan.

These suggestions have been for objective work wherever the subjects admit of objective expression or demonstration. Interest and its applications and average, however, are subjects that are largely informational and do not admit of development in the sense in which the term is used when applied to subjects in which expression may be made in construction language. The outlines in these subjects are *development* plans in that they are organizations of instruction material in natural sequence.

Decimal fractions offer no opportunities for special *objective* work because they do not differ from common fractions, which have had full objective development, in ways that constitute objective variations. The problem in decimal fraction development is not, therefore, that of the nature of a fraction or of the meaning of 10ths, 100ths, or 1000ths. It is the problem of interpreting and expressing values in terms of position with reference to a so called decimal point. The outlines in this subject are development plans which attempt to guide the teacher's work consistently and directly toward this single instruction aim, position language. No other subject in the curriculum of the elementary school offers such possibilities for the formulation and execution of pure, unbroken development plans of instruction.

The outline for percentage and cubic measure are fully objective. Construction work in the former is in the form of diagrams. This is so because it deals with relations—the relations of parts to a whole. No subject in arithmetic offers to the student so many perplexities as this one and yet it lends itself perfectly to objective expression and demonstration in diagram form. There is no condition under the subject that may not be fully expressed by a diagram.

At the conclusion of each subject, short methods of solution are outlined. Definitions of all the principal terms have been suggested. There are rules that every pupil should memorize at the proper time *after* he has mastered the principles or processes which they describe. These rules have been given.

PART III

THE DEVELOPMENT OF DECIMALS

CHAPTER I

THE HISTORY OF INTEGERS

Topics:—Finger Counting. The Abacus With Integral Numbers

A. Aim

Decimals should be taught by repeating objectively in the school room the steps by which integral numbers and then fractions were developed by the race. The child must be taken objectively through (1) the finger counting period, (2) the period of the use of pebbles without the abacus (although this historically is purely conjectural), (3) the period following the invention of the abacus with pebbles, (4) the invention of cones with numerals inscribed on them, (5) the period of discs as substitutes for the cones, and (6) the invention of zero and with this the completion of the integral system. Following this he must be given the experiences in computations which suggested decimal fractions and led to their invention. In this way he acquires a background which enables him to anticipate by inference from integral notation, whose historical development he has traced, the plan for writing 10ths, 100ths, etc.; to feel the need of a sign where we use the decimal point; and to suggest a sign.

B. Finger Counting

1. The Form

Three or more children are selected to stand before the class, facing the class, and express numbers dictated orally by the teacher. The fingers of the child on the right (of those in their seats) express numbers under ten, those of the second child express the 10's, those of the third express the 100's, etc. In expressing numbers the child goes from the little finger of the left hand to the little finger of the right. The little finger of the left hand signifies 1; that, with the next finger expresses 2; etc. When 10 is indicated the child at his right expresses it with his little finger.

The teacher dictates orally, for finger expression, such a series of numbers as 1, 6, 4, 8, 12, 20, 125, 205, 25, 301, 103, 500, 250, 360, 306, etc.

There are two points to be brought out by questions in *every* dictation—that of "used" and that of "empty" counters.

(*a*) THE "USED" COUNTERS

How many persons (savages) are *used*—required to express the number?

(In 1, *one* is *used*. In 8, *one* is *used*. In 12, *two* are *used*. In 250, *three* are *used*. In 205, *three* are *used*.)

If to this question, when used on a number like 250 or like 205, the pupil replies, "Two,"—meaning that since the child at the *right* (or, in case of 205, the 2d child) has no fingers up, he is therefore not *used*—the teacher simply takes this child out of the line. The pupils will then recognize immediately that the number in either case would be 25. The child *was* used—used in the case of 250 to give the other two their proper positions for 10's and 100's; or, in the case of 205, to give the *3d* child his proper position.

What in 205 is the first child used for?

(For units, and also to give the 2d and 3d their proper positions.)

In 250, the 1st child is not used for units but he has *use* in that the 3d and 2d would not have their proper positions without him.

The 3d child is used in 250 or in 205 for *hundreds* only, not to *give position*.

All these matters are of vital importance and must be most carefully worked out.

(*b*) THE "EMPTY COUNTERS"

How many of the *used* children are *empty-handed?*

(In 205, the 2d is empty-handed. In 250, the 1st is empty-handed.)

This step, through the points involved in the questions in (*a*) and (*b*), brings out (1) which of the children are *used;* (2) which are empty-handed; (3) the term *position* (first position, second position, etc.); and (4) what each child (or position) is *used for*—for units or tens or hundreds *only*, to give another child or other children their proper *positions*, or for *both* counting and position uses.

It is very important that these questions be asked with every dictation and the ideas of "*used*" and "*position*" and "*empty-handed*" be driven home.

Many exercises should be given in expressing and reading numbers in finger counting, one group of pupils and then another "acting out" the numbers.

2. *Adding and Multiplying*

Simple exercises in adding and multiplying by "finger" work * should be given. The following expressions indicate the form of the written dictations:

---

* These exercises are not given as illustrations of computations actually required and used by our ancestors of the finger-counting period. They show, rather, the form that computations would have taken under possible conditions. Social development in that age was doubtless not sufficiently advanced to require the use of numbers as high as these in interchanges or

$3 \times 23 =$
$25 + 22 =$
$2 \times 34 =$
$3 \times 22 =$
$51 + 18 =$
$3 \times 5 =$
$5 + 9 =$
$8 + 8 =$
$4 \times 9 =$
$2 \times 28 =$
$3 \times 18 =$
$2 \times 82 =$
$3 \times 73 =$
$18 + 19 =$

C. Pebble Counting

In this step, which, if not established historically, rests upon sound logic, the savage used pebbles instead of fingers and made use of larger pebbles or those of a different color for 10's and 100's, thus doing without the assistance of another person. If he had need of hundreds in his counting, still larger pebbles would serve his purpose for numbers of that order. This step with its small pebbles for units, its larger pebbles for 10's, and its still larger pebbles for 100's is a natural first form of pebble counting—a recourse to which the savage would naturally resort when comrades were not available to assist with their fingers. Again, this form of pebble counting would

---

measurements. That was the childhood of the race. Human wants were few; measuring needs, limited. The aim in these exercises is to make finger-counting methods clear to the pupil, to show what would have been if conditions made such computations necessary.

directly suggest a device like the abacus in which the *position* features of the finger counting were preserved, but with which only one kind of pebbles were required. Again, from finger counting to abacus with pebbles is *two* steps, but the law of development is by *single* steps. For these reasons the conjecture that this form of counting preceded the abacus seems to be sound.

Three sizes of pebbles should be secured for exercises in this step—small objects for units, larger objects for 10's, and still larger objects for 100's. Exercises like those in *B 2* should be given. The questions suggested in *B 1* (*a*) and (*b*) should be used with each construction. The forms of the questions should be changed to, What *positions* (spaces) are used? What *positions* are empty? What is the —— *position* used for? etc.

D. Invention of the Abacus

1. The Abacus Described

The word "invent" should be developed. Our ancestors now "invented" the abacus. A piece of common (wood) board something like 10″ square is brought into class. Sand is sprinkled over it. Grooves are made in the sand by drawing the fingers through it and with these grooves as spaces (positions) numbers are expressed with common pebbles that may be gathered on the street.

The teacher dictates numbers—8, 4, 20, 105, 250, 4005, 25, 108, 180, 1000, 5001, etc.—for one pupil and now another to express in pebble language.

2. *Questions*

The questions suggested in *B 1* (*a*) and (*b*) must be asked continually—

What positions (spaces) are used?

(The spaces, of course, are named from right to left—1st space (or position), 2d space, 3d space, etc. In 123, the 3 is in the first space.)

What positions are empty?

How do you know that the —— position is empty? (The child replies, "I can *see* that it is empty.")

How many positions are used with this number?

3. *Imitation Sandboards*

The sandboard may now be laid aside and each child given a piece of cardboard ruled in imitation of a sandboard. If these substitutes for the abacus are made 18″×4″, there will be room on them for 9 2-inch spaces—three for fractions and 6 for integral numbers. One side of the cardboards should be ruled at this time with 9 spaces in regular form for integers.

When, later, fractions are introduced, the reverse side may be ruled as above for integers *and* fractions. This is to avoid anticipating fractions at this time by ruling for them. If each pupil's card has his name written on it, the cards may be gathered up and redistributed at will and quickly.

Pebbles may be used on these abaci; or, if desired, some other form of object may be substituted—pegs or anything else that is suitable and easily obtainable.

Numbers are dictated orally as for the sandboard above and the question asked as before.

4. Adding, Subtracting, and Multiplying

a. These processes upon the abacus are not unlike the processes with which the child is familiar excepting that they are objective. Our ancestors used them in much the same way as they are used to-day excepting that their multiplication was practically *addition.*

In addition, the numbers are expressed on the abacus and then added, the sum taking the place of the addends at the conclusion of the process.

In subtraction, the *minuend* only is expressed on the abacus and the subtraction performed in a way similar to present day subtraction.

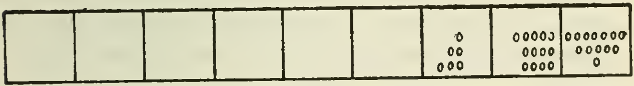
In multiplication, the teacher should have the multiplicand expressed and the work of multiplying performed as we multiply to-day except that the multiplier is not expressed on the abacus.

As to division, the abacus processes, even of the periods hundreds of years after the first use of the abacus, were too complicated and difficult for a child to attempt. Moreover, they would have no development value for him.

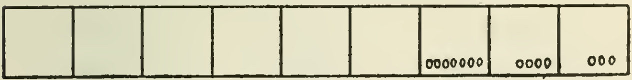
b. Exercises in Adding

325 + 257
508 + 309
1008 + 8007
9046 + 8077
5970 + 6540
380 + 275 + 774

The following diagrams show the expression of 157 + 245 + 341 on the abacus and the sum as it will appear after the work is done:



(1)



(2)

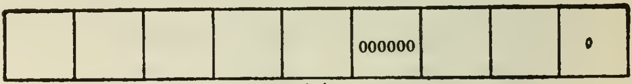
Note 1.—(1) Shows how the child's abacus looks *after* he has expressed the numbers, but before the addition is performed. (2) Shows how his abacus looks after the addition. The sum is 743.

c. Exercises in Subtraction

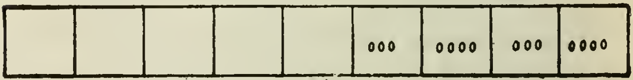
862 − 235
791 − 324
806 − 252
3760 − 1345
901 − 256
560 − 273
8001 − 356
9002 − 3567

The questions in *B 1* (*a*) and (*b*) in the form suggested in *2* above should be used with each.

The following diagrams show (1) the expression of 6001 − 2567 on the abacus and (2) the remainder as it will appear after the construction:



(1)



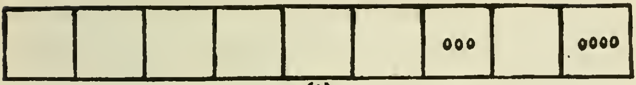
(2)

Note 2.—(1) Shows the abacus after the pupil has expressed the minuend. (2) Shows the abacus after the subtraction has been performed. Remainder, 3434.

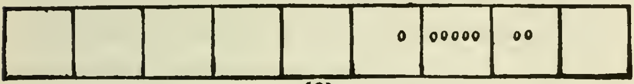
d. Exercises in Multiplication

3 × 261
2 × 426
2 × 528
3 × 461
3 × 518
4 × 354
3 × 507
4 × 450
3 × 5006

The following diagram shows the abacus expression for 5 × 304 ready for the work of multiplying, which, of course, is mental:



(1)



(2)

NOTE 3.—(1) Shows the abacus after the multiplicand has been expressed. (2) Shows the abacus at the close of the construction. The product is 1520.

(Throughout the exercises of this step the questions in *B 1* (*a*) and (*b*) as to *position*, *empty spaces*, and *used spaces* must be continued.)

E. The Invention of "Apices" on Cones

In the third step each unit (of the first or second or third or any other order) had its pebble. Each pebble in the first space had its *real* value, *one*. Each pebble in the 2d space had a representative or position value. It stood for ten units of the first order. In the same way, a pebble in the third space had a position value of 100 units of the first order or ten units of the 2d order.

The third step, therefore, established position or representative values.

The advance in development which features this fourth step is the extension of representative values to *all* abacus counters regardless of the spaces in which they may be used. It is the culmination of the development of the principle of representative values.

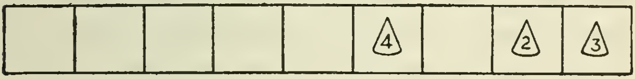
It will doubtless be necessary for the teacher to conduct the work of this step as a class exercise with a large abacus on a table in the front of the room. It will not be possible, ordinarily, to supply the individuals of the class with cones large or small. It may be difficult to procure enough even for general class use. There should be, however, if possible, at least a dozen small ones, two dozen would be much better. If cones are not available, other forms (inch cubes, for illustration) may without great impropriety be used provided that the pupils be taught that the *real* objects were cones. The step is more important *historically* than for development purposes. The cones were soon superseded for greater convenience by discs but the same "apices" were written or engraved on them as were used on the cones.

The pupils are taught that at the close of the pebble-counting period some genius in India invented written signs for 1, 2, 3, 4,—9; that they were brought from India to Alexandria, Egypt, by travelers; that educated men from Rome learned them in Alexandria and taught them to the Romans; that these signs were the ancestors of our figures 1 to 9; that they were used by the Romans by writing them on little cones (some cones with "1," others with "2," others with "3," etc.); that the cone form for counters was chosen because of its stability of equilibrium (it will not tip over easily when used on an abacus); that by their use fewer counters were necessary than with pebbles; and that people called the signs on these counters apices.

Exercises should be given by calling upon one child and then another to express numbers on the abacus from oral dictation.

Exercises for Expression with Cones.

4, 14, 20, 100, 80, 101, 1005, 4023, 5302, 2400, 3000, 4002, 10000, 2020.



(This diagram shows how 4023 was expressed with the abacus.)

The questions suggested in *B 1* (*a*) and (*b*) should be used.

After a few such exercises the teacher passes to the next step.

F. The Substitution of Discs for the Cones

This step differs from the fourth in that (1) discs, like our copper or silver coins, with the number symbols written on them (the same symbols as were used on the cones) are used instead of the cones, and (2) the Roman fractions are introduced.

1. The Discs Described

It will be impossible to supply the pupils with discs for work in this step. Rectangular pieces of cardboard about $\frac{1}{2}''$ square or octagonal pieces of proper size will serve all practical purposes provided that the pupils are taught that the Romans used *coin-shaped* counters, but that not being able to cut them in circular form we use the easier *rectangular* cards. Each pupil should have a supply of such cards —several marked 1, several marked 2, several marked 3, etc., to 9. These figures should be made large, and for convenience for rapid work the figures should be written on both sides of the cards.

2. Exercises with Integral Numbers

Exercises from oral dictation should be given in adding, subtracting, and multiplying.

a. ADDING

The following diagram shows the abacus expressions in the addition of 207 + 328 + 193:

						②		⑦
						③	②	⑧
						①	⑨	③

The following diagram shows the abacus after the addition:

						⑦	②	⑧

Many exercises like the above should be given.

b. Subtracting

The following is the diagram of 820 − 35 before the operation of subtracting:

						⑧	②	

The diagram of the abacus after the operation need not be shown as it corresponds to that in subtraction in pebble counting in *D 4 c* above.

It should be evident that the work of subtracting is not unlike our present subtraction excepting that the subtrahend is not expressed on the abacus but carried in the mind entirely.

Several exercises should be given in subtraction.

c. Multiplying

The following is the diagram of 4 × 823:

						⑧	②	③

The work is mental, the pupil constructing the multiplicand (823) on the abacus as above and then multiplying by 4 in our ordinary way excepting that the multiplier 4 is not expressed. The other (older) form of abacus multiplying would be by expressing the 823 four times and then solving by addition.

The following diagram shows the abacus at the close of the operation:

					③	②	⑨	②

Several exercises should be given in multiplying with the abacus.

SUMMARY.

1. The method of expressing number in the "finger counting" age. "Positions" indicated by persons; units, by fingers.
    1st. How it required a separate individual for the tally of the units of each order. (*B, 1.*)
    2d. How the individual's "position" would necessarily be regarded as "used." (*B, 2.*)
    3d. How the individual's "position" would be "empty" yet "used." (*B, 3.*)

4th. Exercises in "finger" addition and multiplication. (*B*, *4*.)

2. Pebble counting—"positions" indicated by *sizes* of pebbles; units, by pebbles of the different sizes. (*C*.)
3. The invention of the abacus—"positions" indicated by spaces on the sandboard; units, by pebbles.

   The beginnings of position values.

   The abacus described. (*D*, *1*.)

   Exercises in addition, subtraction, and multiplication. (*D*, *4*.)
4. Written number signs (apices) on cones—

   The establishment of position values. (*E*.)

   Exercises in expressing numbers.
5. The substitution of coin-shaped counters for the cones—discs inscribed with the number signs, apices. (*F*.)

   The discs described. (*F*, *1*.)

   Exercises in addition, subtraction, and multiplication. (*F*, *2*.)

Time required for the work of this chapter, twenty days.

CHAPTER II

THE ROMAN FRACTIONS

Topic:—The Notation and the Operations on the Abacus

A. Roman Notation

1. Integers

It is necessary for us before taking up the last step in the development of integral number—the introduction of zero—to develop the Roman system of fractions.

The teacher will bear in mind that the abacus was the instrument of Roman *calculation* but not that for expressing *written* number. Numbers were *written* in what we now call Roman numerals. 534 was DXXXIV, 1667 was MDCLXVII, 249 was CCXLIX. In adding, subtracting, multiplying, and dividing, the computations were made on the abacus with the *apices* as shown in Chapter 1; and the results, if written down, were written in Roman numerals. The Roman *fractions*, when written, were expressed in the same way and took our present fraction form—the numerator over the denominator with a horizontal line between. $\frac{5}{12}$ was $\frac{\mathrm{V}}{\mathrm{XII}}$, 10 was $\frac{\mathrm{X}}{\mathrm{XII}}$, etc.

The aim of this chapter is to show how fractions were expressed on the abacus and how fractions on other bases than 12 would have been expressed on the same principle. With these experiences in using fractions of several bases, the pupil, when the final step in number development is taken—the invention of zero—and we suggest the expression of 10ths and 100ths with the zero and *without* the abacus, will be ready to write and read them by unhesitating inference.

2. *Fractions*

As has been said, the fractions that the Romans used in computations were based upon 12—12ths, 144ths, 1728ths, etc. These fractions were used on the abacus in spaces below (to the right of) the spaces for integers. $\frac{1}{2}$ was expressed by VI in the first fraction space ($\frac{6}{12}$), $\frac{1}{3}$ by IV, $\frac{1}{4}$ by III. X in the 2d fraction space meant $\frac{10}{12}$ of $\frac{1}{12}$ or $\frac{10}{144}$. $\frac{1}{8}$ was expressed by I in the first fraction space and VI in the 2d fraction space. Their reduction of $\frac{1}{8}$ was somewhat as follows: $\frac{1}{8}$ of $\frac{12}{12} = \frac{1\frac{1}{2}}{12}$. This gives I in the first fraction space. $\frac{\frac{1}{2}}{12} = \frac{1}{2}$ of $\frac{1}{12} = \frac{1}{2}$ of $\frac{12}{144} = \frac{6}{144}$ or VI in the 2d space. Their reduction of $\frac{1}{9}$ was after the same manner—$\frac{1}{9}$ of $\frac{12}{12} = \frac{1\frac{1}{3}}{12}$. This gave I in the first fraction space. $\frac{\frac{1}{3}}{12} = \frac{1}{3}$ of $\frac{1}{12} = \frac{1}{3}$ of $\frac{12}{144} = \frac{4}{144}$. This gave IV in the 2d fraction space.

In this way all fractions other than 12ths were changed to the 12ths series and expressed, in making computations, in the fraction spaces of the abacus. Exercises should be given to teach the pupils this process of changing other fractions to duo-decimals.

The teacher will bear in mind that at this time (the period in which the apices were on discs) the "0" was not known in Rome, neither the position system of the Arabs. On this account apices could not be used for fractions. $\frac{10}{12}$ could not be expressed except by $\frac{\text{X}}{\text{XII}}$, or by X in the first fraction space, because neither 10 nor 12 could be expressed with Arabic symbols (apices) in a single abacus space. The same would be true of $\frac{11}{12}$. It could be expressed only by Roman letters $\frac{\text{XI}}{\text{XII}}$ or by XI in the first fraction space. For these reasons the Romans used the Roman numerals on the abacus when expressing fractions. A mixed number on the abacus would show apices in the integral spaces and Roman numerals in the fraction spaces.

NOTE.—As our decimal fraction system developed from these Roman "duodecimals," it is important that the latter should be understood fully. To this end the duodecimals should be written out in *table* form and pupils should be able to give this table orally:

12 ′′′′′ = 1 ′′′′
12 ′′′′ = 1 ′′′
12 ′′′ = 1 ′′
12 ′′ = 1 ′
12 ′ = 1 unit (whole)
10 units = 1 ten
10 tens = 1 hundred, etc.

This shows the Roman number table from the 5th fraction order to the 3d integer order.

For convenient oral expression the above tables may be read—

12 fives = 1 four
12 fours = 1 third
12 thirds = 1 second
12 seconds = 1 prime
12 primes = 1 integral unit
10 integral units = 1 ten
10 tens = 1 hundred

The table; exercises in expressing fractions and mixed numbers on the abacus; and exercises in addition, subtraction, and multiplication, will give the pupils the necessary familiarity with the system.

B. The Abacus With Fractions

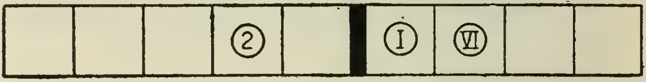
1. *The Discs*

For work with fractions, therefore, it will be necessary to prepare sets of counters with the Roman letter numbers—

these counters (discs if possible) to be used in the fraction spaces, the apices in the integer spaces.

2. *The Fraction Spaces*

The form of the abacus must now be changed to provide fraction spaces. Its form should be as follows, the wide space between grooves being the space between integral and fractional numbers:



(The teacher should ask frequently what this wide space between the integer and fraction spaces is. The pupils will call it "the stopping place of integers." This should be satisfactory.)

This abacus diagram shows the abacus expressing $20\frac{1}{8}$.

3. *Exercises in Expressing Fractions*

A. EXERCISES WITHOUT REDUCTIONS

Exercises like the following should be dictated orally for pupils to express on the abacus: $\frac{8}{144}$, $\frac{10}{1728}$, $\frac{9}{12}$, $\frac{7}{144}$, $\frac{18}{144}$, $\frac{25}{1728}$, $\frac{35}{12}$, $\frac{19}{144}$, $\frac{65}{12}$, $\frac{29}{1728}$, $\frac{97}{144}$, $\frac{29}{144}$, $\frac{110}{1728}$, $\frac{250}{1728}$, $\frac{39}{144}$, $\frac{39}{12}$, $\frac{44}{144}$, $\frac{109}{144}$, $\frac{39}{1728}$, etc.

NOTE.—In $\frac{39}{144}$, the pupil expresses it as $\frac{3}{12}$, $\frac{3}{144}$. In $\frac{109}{144}$ the pupil constructs $\frac{9}{12}$, $\frac{1}{144}$. $\frac{65}{12}$ is $5\frac{3}{12}$. The aim of these exercises is, in this way, to give the pupil a practical insight into the fraction scheme. This form of expression will be particularly valuable in the decimal fraction work later.

B. EXERCISES WITH REDUCTIONS

$31\frac{1}{4}$	$96\frac{3}{4}$	$36\frac{7}{8}$
$108\frac{2}{3}$	$19\frac{4}{18}$	$21\frac{1}{6}$
$200\frac{5}{6}$	$39\frac{1}{9}$	$24\frac{1}{16}$
$310\frac{1}{2}$	$58\frac{4}{9}$	$35\frac{1}{18}$

C. Reductions of Fractions

1. *Forms of Reductions*

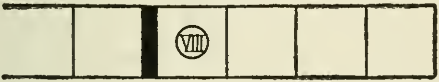
The pupils must be taught to make reductions to duodecimals in two ways, both of which will be required in reducing common fractions to decimals when the latter are introduced.

a. THE LONG WAY

(i) $\frac{1}{4} = \frac{1}{4}$ of $1 = \frac{1}{4}$ of $\frac{12}{12} = \frac{3}{12}$.

We now put III in first fraction space of the abacus.

(ii) $\frac{2}{3} = \frac{2}{3}$ of $1 = \frac{2}{3}$ of $\frac{12}{12} = \frac{8}{12} =$ VIII in first fraction space. (See diagram (1) below.)



(1)

(iii) $\frac{1}{10} = \frac{1}{10}$ of $1 = \frac{1}{10}$ of $\frac{12}{12} = \frac{\frac{12}{10}}{12} =$ I in first space. $\frac{\frac{2}{10}}{12} = \frac{2}{10}$ of $\frac{1}{12} = \frac{2}{10}$ of $\frac{12}{144} = \frac{1}{10}$ of $\frac{24}{144} = \frac{\frac{44}{10}}{144} =$ II in second space.

$\frac{\frac{4}{10}}{144} = \frac{4}{10}$ of $\frac{1}{144} = \frac{4}{10}$ of $\frac{12}{1728} = \frac{1}{10}$ of $\frac{48}{1728} = \frac{\frac{48}{10}}{1728}$.* (See diagram (2.)

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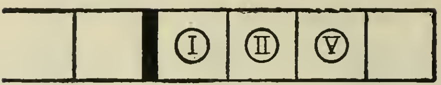
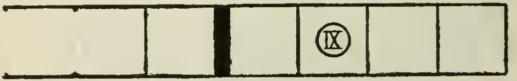
* This is approximately V in the third fraction space.



(2)

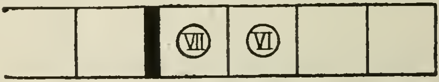
NOTE 1.—The terms approximate and approximately must be developed and must be used very frequently so that the pupil, in the case of an indeterminate duodecimal fraction like that for $\frac{1}{10}$ as above, may learn to say in stating the result of the reductions, "$\frac{1}{10}$ is approximately $\frac{1}{12}$, $\frac{2}{144}$, $\frac{5}{1728}$." It will be necessary to dictate a reduction, "Give the approximate of $\frac{3}{5}$ to the third term of the duodecimal series." "Express the 4th term approximate of $\frac{2}{7}$." The child will learn to say of a reduction, "it is exact"; or "it is approximate." These same terms will be needed in decimals later.

(iv) $\frac{1}{16} = \frac{1}{16}$ of 1 $= \frac{1}{16}$ of $\frac{144}{144}$ * $= \frac{9}{144} =$ IX in 2d space. (See diagram (3).)



(3)

(v) $\frac{5}{8} = \frac{5}{8}$ of 1 $= \frac{5}{8}$ of $\frac{12}{12} = \frac{1}{8}$ of $\frac{60}{12} = 7\frac{\frac{4}{8}}{12} =$ VII in 1st space. $\frac{\frac{4}{8}}{12} = \frac{4}{8}$ of $\frac{1}{12} = \frac{4}{8}$ of $\frac{12}{144} = \frac{1}{8}$ of $\frac{48}{144} = \frac{6}{144} =$ VI in 2d space. (See diagram (4).)



(4)

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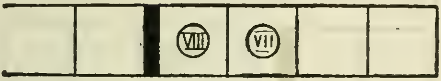
* $\frac{1}{16}$ being smaller than $\frac{1}{12}$ cannot be made into units of the first order (12ths). This the child will observe. He passes to units of the 2d term of the series 144ths).

(vi) $\frac{5}{7} = \frac{5}{7}$ of $1 = \frac{1}{7}$ of $\frac{60}{12} = \frac{8\frac{4}{7}}{12} =$ VIII in the 1st space. $\frac{\frac{4}{7}}{12} = \frac{4}{7}$ of $\frac{1}{12} = \frac{4}{7}$ of $\frac{12}{144} = \frac{1}{7}$ of $\frac{48}{144} = \frac{6\frac{6}{7}}{12} =$ VII in 2d space approximately. (See diagram (5).)



(5)

(This "long way" will be needed in decimal fractions in extending mixed decimals. $.3\frac{4}{5}$, for illustration—$\frac{\frac{4}{5}}{10} = \frac{4}{5}$ of $\frac{1}{10} = \frac{4}{5}$ of $\frac{10}{100} = \frac{1}{5}$ of $\frac{40}{100} = \frac{8}{100} = .08$. $.3\frac{4}{5} = .38$).

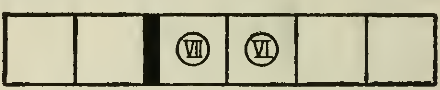
b. THE SHORT WAY

This short way is the *regular* process of reducing common fractions to duodecimals and it will be the regular way of reducing common fractions to decimals when we reach that subject.

NOTE 2.—In this form of reductions the dictations are given with instructions definitely limiting the number of the term to which a reduction is to be carried. The direction "exact or to the 4th term" would mean that in case the duodecimal does not terminate in or before that term the pupil will give the approximate series to the 4th term. There is nothing gained in this development work by carrying reductions beyond the 3d or 4th term of the series.

In preparing to make the reduction, the pupil must first find where the duodecimal will terminate. He then changes to that term direct.

(i) $\frac{5}{8}$, exact or to the 3rd term. $\frac{5}{8}=\frac{5}{8}$ of $1 =\frac{5}{8}$ of $\frac{144}{144}=\frac{1}{8}$ of $\frac{720}{144}=\frac{90}{144}$.



The pupil by trying finds that 8ths could be changed to 144ths. He then proceeds to change to 144ths. He finds that there are 90. $\frac{90}{144}=\frac{8}{12}+\frac{6}{144}$.

(ii) $\frac{4}{5}$, exact or to the 3rd term approximate. $\frac{4}{5}=\frac{4}{5}$ of $1 =\frac{4}{5}$ of $\frac{1728}{1728}=\frac{1}{5}$ of $\frac{6912}{1728}=\frac{1382\frac{2}{5}}{1728}=$ $9' \; 7'' \; 2'''$ approximate.

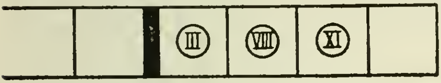
The pupil changes the 1728th (‴) to 144ths (″) and the 144ths to 12ths (′).

NOTE 3.—This will be the *regular* process of reductions in decimals when we reach them. This process applied to reductions in decimals later is shown in the following—$\frac{5}{8}=\frac{5}{8}$ of $1=\frac{5}{8}$ of $\frac{1000}{1000}=\frac{1}{8}$ of $\frac{5000}{1000}=\frac{625}{1000}=.625$. This being the case this must be used very many times for the sake of the analytical process.

2. *Questions to be Used*

The question as to position (first integral, first fractional, second fractional, second integral, etc.), number of used spaces (positions), and number of empty spaces (positions) must be in constant use. The question, "What is this?" must be applied again and again to the wide division line between the integral and fraction spaces. "It is the place where integers stop and fractions begin." "It is the stopping place for integers."

Note.—*Reading Duodecimals.*—In reading duodecimal fractions, the pupil calls them 12ths, 144ths, 1728ths, etc.



would be read $\frac{3}{12}$, $\frac{8}{144}$, $\frac{11}{1728}$.

The teacher in dictating them for expression on the abacus would give them these names.

3. *Exercises for Reductions*

a. Long Way

Change to duodecimals exact, or approximate to three terms—$\frac{3}{5}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{8}{9}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{5}{6}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{3}{7}$, $\frac{5}{8}$, $\frac{5}{9}$.

b. Short Way

Change to duodecimals exact or approximate to three terms—$\frac{4}{5}$, $\frac{5}{11}$, $\frac{5}{14}$, $\frac{7}{16}$, $\frac{9}{10}$, $\frac{7}{8}$, $\frac{5}{6}$, $\frac{5}{9}$, $\frac{4}{6}$, $\frac{15}{16}$, $\frac{11}{15}$, $\frac{8}{10}$, $\frac{4}{9}$.

D. Operations on the Abacus

1. *Addition*

$20\frac{5}{6}$	$2070\frac{7}{8}$	$315\frac{1}{6}$	$6009\frac{5}{8}$
$206\frac{5}{9}$	$3006\frac{2}{3}$	$293\frac{7}{8}$	$3907\frac{23}{24}$
$375\frac{3}{4}$	$5178\frac{15}{16}$	$309\frac{1}{24}$	$5093\frac{1}{4}$

$35\ \frac{8}{12}\ \frac{7}{144}\ \frac{6}{1728}$

$203\ \frac{9}{12}\ \frac{8}{144}\ \frac{10}{1728}$

$250\ \frac{7}{144}\ \frac{8}{1728}$

$272\ \frac{3}{12}\ \frac{4}{1728}$

$200\ \frac{7}{12}\ \frac{8}{144}$

$250\ \frac{8}{12}\ \frac{9}{144}\ \frac{10}{1728}$

Many other similar exercises should be given.

Note.—There are two ways in which dictations of duodecimal fractions for abacus work may be written—(1) with the denominators as in $\frac{3}{12}$, $\frac{4}{144}$, $\frac{5}{1728}$; and (2) with the signs $'$ $''$ $'''$ $''''$, etc., as in $3'$, $4''$, $5'''$, and $6''''$. There are also two ways of reading such fractions—(1) with the denominators, as in the example above—3 twelfths, 4 one hundred forty-fourths, etc.; and without the denominators, as 3 primes, 4 seconds, 5 thirds, etc.

The pupils, however, should be required to read *with the denominators* almost entirely—only occasional readings being permitted in the prime, second, third, etc. language. The habit of reading *with denominators* is *essential* in decimal fraction development.

Besides being essential, the reading with denominators is the only reading that is expressive of anything that is *definite*. $36\ 4'\ 5''\ 6'''$ means 36 and $\frac{10}{12}$, $\frac{5}{144}$, $\frac{6}{1728}$; or 36 and $\frac{4}{8}$, $\frac{5}{64}$, $\frac{6}{512}$; or 36 and $\frac{4}{10}$, $\frac{5}{100}$, $\frac{6}{1000}$, according as the fraction basis is 12, 8, or 10. If the fraction basis is 6, 36 and $4'\ 4''\ 5'''$ would mean 36 and $\frac{4}{6}$, $\frac{4}{36}$, $\frac{5}{216}$. If the fraction basis is 9, the above expression would mean 36 and $\frac{4}{9}$, $\frac{4}{81}$, $\frac{5}{729}$. Whatever the system, the reading *without the denominator* would be in terms of primes, seconds, thirds, etc. As this outline goes (later) from one fraction basis to another (12's, 8's, 10's, 6's, etc.) the prime-second-third reading should be used with each system just enough to establish in the child's mind that that language applies, whatever the system—that the signs $'$ $''$ $'''$ and the prime-second-third language refer to *fractional positions* (first, second, third, etc.) regardless of whether the system is based on 12 or 10 or 8 or 6 or some other number. As a matter of historical fact, the prime-second-third, etc. signs were used with decimal fractions ($853\ 1'\ 2''\ 5'''$ meaning 853 and $\frac{1}{10}$, $\frac{2}{100}$, $\frac{5}{1000}$; or, in our language, 253.125) in the period between duodecimals and the discovery and adoption of our present position system with decimals. It was the intervening step between the one and the other. (See Fink, p. 50 and Ball, p. 203.)

2. *Exercises in Subtraction*

$$\begin{array}{ccc} 300\frac{3}{8} & 406\frac{1}{4} & 7003\frac{1}{6} \\ -201\frac{11}{16} & -183\frac{5}{9} & -2124\frac{7}{9} \\ \hline \end{array}$$

From $255\ 6'\ 2''\ 5'''$ take $17\ 5'\ 8''\ 2'''$
From $790\ 3'\ 10''\ 2'''$ take $201\ 5'\ 8''\ 5'''$
From $900\ 2'\ 2''\ 2'''$ take $101\ 6'\ 6''\ 6'''$

Several exercises in subtraction should be given.

3. *Exercises in Multiplication*

Multiply $34\ 2'\ 8''\ 5'''$ by 5
Multiply $106\ 8'\ 8''\ 8'''$ by 3
Multiply $2005\ 6'\ 1''\ 6'''$ by 7

Many other exercises in multiplication should be given.

4. *Division*

The operation of division of integral numbers on the abacus was very complicated and difficult. On this account it has no development value and should not be attempted. The other processes have served every purpose of leading the child over the path of the evolution of decimals from the Roman fractions. Division, if it could be taught, would add nothing in the way of clearer or more complete imagery.

SUMMARY.

1. How the Romans *expressed* integral numbers without the use of the abacus, using Roman numerals. (*A*, *1*.)
2. How in their computations the Romans used the abacus with the apices. (*A*, *1*.)
3. The Roman fraction notation was in Roman numerals and was based on 12. Each denominator was some power of 12. (*A*, *2*.)
4. Roman calculation required the abacus with apices for integers and Roman discs, or the position signs (′ ″ ‴ etc.), for fractions. (*B* and *D* note.)
5. Methods of reducing other fractions to the Roman duodecimal basis for expression and calculation. (*C*, *1*.)

The "long way" method. (*C, 1 a.*)
The "short way" method. (*C, 1 b.*)
How to question. (*C, 2*). How to read fractions. (*C, 2* Note.)

6. Operations in addition, subtraction, and multiplication on the abacus with Roman fractions. (*D.*)

Time required for the work of this chapter with necessary reviews of previous work, Chapter I, twenty days.

CHAPTER III

FRACTIONS HAVING OTHER BASES, DUODECIMAL IN FORM

Topic:—If the Romans Had Used 6 or 8 or 10, Etc., as a Fraction Basis

A. Aim

To introduce other fraction bases with exercises in reading and writing, addition, subtraction, and multiplication under each. Experience with 8, 6, 10, 4, etc., as bases, will give the child (1) better insight into fractions of this kind, and (2) an opportunity to discover, without its being suggested to him, that 10 is the most workable basis for fractions in abacus computations. The men who discovered decimals discovered first the superior workableness on the abacus of fractions of the 10's basis; then, their adaptation to the position system with 0. They discovered that it is easier and that there is less likelihood of errors to multiply 84 and $2' \ 5'' \ 6'''$ by 9 when that expression means 84 and $\frac{2}{10}$, $\frac{5}{100}$, $\frac{6}{1000}$, than when it means 84 and $\frac{2}{12}$, $\frac{8}{144}$, $\frac{6}{1728}$ or 84 and $\frac{2}{8}$, $\frac{5}{64}$, $\frac{6}{512}$. The aim of the step is to give the pupil experience in rapid work in addition, subtraction, and multiplication under now one, then another, and then another fraction basis. In this way he, too, will discover the facility with which fractions on the 10's basis are handled. He will prefer to use them, to write and read them, and to compute with them. From this to real decimal fractions is but a step.

B. Fractions on Various Bases

1. On 8 Basis

a. Introduction.

This is introduced by asking the question, "If the Romans had used 8 as their fraction basis instead of 12, what would have been the denominators of their fraction series?" It is usually found that the pupils will respond at once, "8ths, 64ths, 512ths, etc."

b. Exercises in Expressing Fractions.

In expressing these numbers, fractional as well as mixed, on the abacus, the pupils should use the discs with the apices. There is no necessity for discs having the Roman letter numerals (I, II, III, IV, etc.) with fractions on a basis *less* than *11*. The Romans used them because there were no apices for 10 and 11.

Express 2 and $\frac{7}{8}$, $\frac{5}{64}$.
19 and $\frac{3}{8}$, $\frac{7}{64}$, $\frac{5}{512}$.
20 and $\frac{5}{64}$.
305 and $\frac{2}{64}$, $\frac{4}{512}$.

(The questions as, How many spaces are used? How many are empty? How many *fraction* spaces used and how many empty? How many *integral* spaces used and how many empty? and, What does the wide division line on the abacus mean? should be in constant use as before.)

Express $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{3}{2}$, $\frac{11}{8}$, $\frac{1}{16}$, $\frac{7}{16}$.

Express exactly or approximately to three terms—

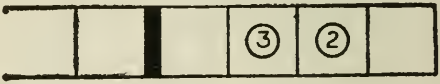
$\frac{5}{8}$, $\frac{7}{16}$, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{5}{9}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{5}{12}$, $\frac{5}{7}$, $\frac{7}{8}$, $\frac{4}{5}$.

The process of changing $\frac{1}{20}$ to fractions on this 8 basis is like that used with 12ths, 144ths, etc.—

$\frac{1}{20} = \frac{1}{20}$ of $\frac{64}{64} = \frac{3\frac{4}{20}}{64}$. We put a 3-disc in the second fraction space.

$\frac{\frac{4}{20}}{64} = \frac{4}{20}$ of $\frac{1}{64} = \frac{4}{20}$ of $\frac{8}{512} = \frac{1\frac{12}{20}}{512} = \frac{2}{512}$ approximately.

We put the 2-disc in the third abacus space for the approximate result.



Many exercises in expressing fractions on this basis should be given.

c. Exercises in reading fractions with denominators when dictated in terms of primes, seconds, etc. (31 4′ 6″ 2‴).

d. Exercises in Addition, Subtraction, and Multiplication.

No suggestions are needed here.

This is the same kind of work as was done with 12ths, 144ths, etc.

2. On 6 Basis

This repeats with 6ths, 36ths, 216ths, etc., the work that was done with fractions on the 8 basis in *1* above.

3. On 10 Basis

This repeats the work that was done with fractions on the 12, 8, and 6 bases above.

4. Mixed Exercises

a. Under this head the teacher gives exercises (1) in addition—adding now an exercise with fractions on a 12 basis, then one on an 8 basis, then one on a 10 basis, then one on a 12 basis, then one on a 6 basis, no two successive exercises on the same basis; (2) in multiplication, going from one basis to another in the same manner.

The child should solve each problem as *rapidly* as possible consistent with accuracy.

Very many exercises in each of the two processes (addition and multiplication) should be given.

b. The aim of the work is to give the pupil an opportunity to discover that fractions on the 10 basis are very much easier to use in computations, easier and with less tendency to error.

c. Comparing the Different Bases.

Our forefathers made this discovery and from it came eventually the application of our integral position system to fractions on a 10 basis—our present decimal fractions. After several exercises have been given, the teacher should seek the opinions of individual pupils as to which of the fraction bases is the easiest to work with. The choice will fall to fractions on the 10 basis.

d. Choosing a Basis for Future Use.

The pupils should now be told that our forefathers made the same discovery and actually changed their fraction system from the 12 to the 10 basis form. It should now be understood by the pupils that, in their work in writing fractions on the Roman plan, in the absence of specific directions to use the 12 basis, they must use this simpler and better 10 form.

SUMMARY.

1. The aim in this chapter is to give the pupil such familiarity with the Roman fraction scheme as will prepare him to

discover for himself in the next chapter our present system of decimal fraction notation. It aims to do this by giving him experience—

(1) In working out fraction systems on the Roman model using various bases—with 8 as a basis, with 6, with 9, but particularly with 10.

(2) In computing with each system thus worked out, to enable him to discover the superiority for computation work of the system on the 10's basis over that on any other basis.

2. Exercises in writing fractions based on the powers of 8, on the powers of 6, on the powers of 10. (*B*, *1–3*.)

Reductions and computations under each of these systems. (*B*, *4*.)

Opinions as to the comparative workableness of these Roman-model systems. (*B*, *4*.)

Time required for the work of this chapter, including necessary reviews of the subject, ten days.

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CHAPTER IV

THE PASSING OF THE ABACUS—ZERO INVENTED

Topics:—The History of Zero With Integral Numbers. The Use of Zero With Fractions. The Need of a Decimal Point. The Period Adopted

A. The Historical Connections

We have brought the development work up to the point where the child, as was the race, is ready for the discovery of the zero and the position system *in both integers and fractions.* Historically, as has been stated, the application of zero and the position system to *fractions*, which gave us our decimal fraction system, was 150 years after the use in Europe of zero with integers.

For development work this difference of time is ignored and the work proceeds on the basis of the full introduction of zero (with both integers and fractions) at one time. That 150 years was a period of assimilation. It recorded no step in advance, hence furnished no element to represent it in the development sequence.

The zero was not invented in Europe and consequently we must drop the history of *Roman* number, go back in time 1000 years, find our zero and the position system where it originated (in India), and trace its connection with the Roman number system. The apices as has been stated were brought from India to Alexandria and thence to Rome. This was three or four hundred years before the zero was invented.

We owe our knowledge of this sign and the principle of position to the Saracen conquests. They were brought by these people to Northern Africa and to Europe through Spain. In France, the Netherlands, and Italy, Arabic and Roman number met. The Arabic system of integral number with its zero applied to fractions as well as to whole numbers, and the Roman system of fractions, changed from a basis of 12 to a basis of 10, became the European system.

The pupils should know these facts—where zero was invented and how it came to Europe —but the development work must proceed as if the zero were an invention of the period when the "Arabic" system practically supplanted that of Rome.

B. The Introduction of Zero

1. With Integers

The teacher tells how some ingenious Indian invented a sign for "empty." This sign was our zero. He told people that if they would use his zero (0) there would be no need of an abacus and discs. He told them that under his new plan (without an abacus), when one writes a number (5, for illustration), it must be understood to be in the first integral space. 4 written alone must be understood to be four. If we wish to write 5 in the second integral space we must say that the first space is empty —50. The expression 300 means the first and second integral spaces empty and the 3 in the third space. The number is three hundred. There are three *used* spaces and one of these, the 3d, is occupied.

"Write a number having 4 in the third and 3 in the first integral space." ("403.")

"What is the number?"

"How many empty spaces?"

"How many used spaces?" ("Three.")

"Write 5 in the fourth integral space." ("5000.")

"What does it say as to used and empty spaces?"

("It says that there are 4 used spaces, the 4th space occupied, the 1st, 2d, and 3d empty.")

"Why the zeros?"

"Read the number." ("Five thousand.")

Several additional exercises with integral numbers should be given and the pupils asked about the used, the occupied, and the empty spaces, and why the zero; as—"Write a number having 6 in the 4th integral space and 5 in the 2d." ("6050.")

2. *With Fractions*

"Write a number having 6 in the first integral space and 4 in the first fraction space." ("6|4."*) (See Chapter III, *B*, *4 d.*)

"Yes, this is good."

"How many used spaces?" ("Two.")

"How many integral spaces used?"

"How many fraction spaces used?"

"What is the vertical line for?" ("To show where integers stop and fractions begin.") (See Chapter II, *C 2.*")

"Read it." ("Six and four tenths.")

"Write the number in the regular way of writing numbers." ("$6\frac{4}{10}$.")

"Write a number having 4 in the 2d integral space, 4 in the 1st fraction space, and 5 in the 2d fraction space." ("40|45.")

(The questions as to the *used* spaces, the *occupied* spaces, the used *integral* spaces, the used *fraction* spaces, and what the verticle line is for should be asked.)

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* The child will certainly write the number and will write it as above. The vertical line is the only means of separating integral from fraction spaces that he has known. It will occur to him here that such separation must be made and he will indicate it as it was indicated on his abacus. The teacher must permit the use of the line in several expressions. She then teaches the simpler "stopping-place" sign—the period.

"Read the number." (Forty and four tenths and five hundredths." *)

"Write six tenths." ("|6").

"Write 8 tenths and 9 hundredths." ("|89.")

"Write 3 and 7 hundredths." ("3|07.")

"How many used fraction spaces?"

"How many empty fraction spaces?"

"What fraction space is empty?"

"What fraction space is occupied?"

"What is the vertical line for?"

"Write 3 hundredths and 4 thousandths." ("|034.")

(Ask with this, and with every other exercise, about the used, empty, and occupied spaces and the vertical line.)

C. The Stopping Place Sign—Written

1. The Period Introduced

The pupils are now taught that our forefathers agreed that the sign for the "stopping place" for integers should be the period. So, instead of the vertical line, we will use that sign hereafter.

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* This "long way" of reading fractions is the child's natural way of reading—natural because he so read fractions on the abacus. The "short way" of reading fractions is a later development. (See *C*, *5*, below.) In the meantime the teacher must not permit herself to read a fraction in other than this "long way."

2. *Written Dictations for Pupils to Read*

.3, 1.4, 30.04, 3.125, 14.006, 1.012, 3.101, 4.002, 1.123, 3.0102, 7.7071, etc. Many such expressions should be dictated.

The pupils should read these fractions "the long way."

Note.—Questions to be used constantly—

(*a*) "What does the period mean?" ("It shows the stopping place of integers. It separates the integral from the fraction spaces.")

(*b*) "In what space is the 3?" ("In the first fraction space.")

"How many used spaces?" ("One.")

"How many occupied spaces?"

(*c*) "Where do we always write tenths?"

"Where do we always write hundredths?"

"Where do we always write thousandths?"

(*d*) "When I write a number in the 2d fraction space what is its value?" ("Hundredths.")

"When I write a number in the 3d fraction space what is its value?" "In the 1st fraction space?" "In the 4th fraction space?"

(*e*) "What is the 1st term of the 10's series of fractions?" "The 3d term?" "The 2d?" "The 4th?"

The questions in (*a*) and (*b*) apply to all dictations for reading or writing mixed or fractional numbers. Those in (*c*) (*d*) and (*e*) may be asked at any time and should be used very many times.

3. *Oral Dictations for Pupils to Write*

3 tenths, 9 tenths, 6 tenths, 4 hundredths, 8 hundredths, 1 hundredth, 8 thousandths,

5 thousandths, 3 ten-thousandths, 6 tenths, 9 tenths, 12 tenths, 18 tenths, 25 tenths, 46 tenths, 8 hundredths, 12 hundredths, 24 hundredths, 141 hundredths, 6 thousandths, 95 thousandths, 251 thousandths, 4 thousandths, 422 thousandths, 15 ten-thousandths, 224 tenths, 735 hundredths, etc.

Many exercises should be given in writing fractions of these kinds from oral dictation and the exercises should be given frequently later.

The teacher will understand that when 12 tenths is dictated for the first time as above, no help or suggestions are allowable. The teacher may ask the questions in (*c*) and (*d*) in Note under *2* above when a pupil hesitates over 12 tenths or 18 tenths or 141 hundredths or 251 thousandths These questions are enough. Give the pupil *time*. With these before him, he will overcome any difficulties that such dictations present. This form of exercise is of great value in establishing the place value in the pupil's mind. Many dictations should be given at this time and similar exercises frequently in the future.

4. Written Dictations for Pupils to Read—"Long Way." (*Footnote p. 528.*)

.45, .62, .125, .283, 12758, .675, .842, .1756, .2345, .18, .182, 1.8, 2.75, 3.82, 5.16, 3.8, 7.52, 38.1, 27.9, etc.

5. The "Short Way" of Reading Decimals. The teacher writes on the blackboard a fraction as—

a. .35.

"Read it." (The pupil reads it as $\frac{3}{10}$ and $\frac{5}{100}$.)

"How many fractions?" ("Two.")

"The denominator of the first?" (10ths.")

"The denominator of the 2d?" ("100ths.")

"What is the lowest common denominator?" ("100ths.")

"Read the expression in this common denominator." (The pupil will read it as 100ths —thirty-five hundredths.)

"Read it as fractional units of one kind." ("Thirty-five hundredths.") "This is the short way."

b. .47. "Read it."

"How many fractions?"

"The denominator of the 1st?" "Of the 2d?"

"The lowest common denominator?"

"Read it in this common denominator."

The teacher should very frequently speak of this as the "short way" of reading the fraction.

c. .023.

The teacher asks in the same way for pupils to read it, to state the number of fractions, 2, to state the lowest common denominator

(1000ths), to read it in that denominator, and to read it as fractional units of one kind.

d. Other similar exercises with .29, .065, .203, .024, .307, .0025, .1004, .0106, .01002, .1003, .256, .167,.0142, .01023, etc.

In each of these the pupil is required to read the expression the long way, to state the number of fractions, to name the several denominators, to give the lowest common denominator, and to read the expression in its lowest common denominator—the short way.

Note.—Hereafter, the teacher will keep in use, in connection with reading decimals, the terms "long way," "short way," "in the common denominator," and "as fractional units of one kind." It should be unnecessary to suggest that frequent exercises in writing decimal fractions from oral dictations and reading written exercises are of highest importance.

D. The " Stopping-Place " Sign—Oral

This section aims to train the child in the proper use of "*and*" in reading decimal fractions and mixed numbers. This is the *oral* stopping-place sign.

1. Written Exercises for Reading

25.9	412.31	32.0302	12.0101	1.1091
3.256	467.67	15.001	2.0225	3.3003
81.18	21.0021	101.102	102.201	4.0004
.171	.0012	.356	.5786	.1111

Special attention must be given to the "and" between the integral and fractional parts of mixed numbers. Such attention to "*and*" and its *banishment from every other place* is a very important feature of these exercises. Very many exercises should be prepared under this head.

2. Oral Exercises for Writing

Exercises like those in (*1*) should be dictated orally by the teacher for pupils to write. In these exercises further opportunity offers for giving experience in the proper use and interpretation of "and." The teacher must be especially careful not to use "and" except at the "stopping place."

3. Exercises for Writing in Words

These are exercises like those in (*1*) for pupils to *write out* in words. "*Written out*" exercises of this kind test the pupil's knowledge of the proper use of "and" in reading and writing numbers.

4. Exercises in "Written Out" Numbers

These are exercises in which the dictations, fractions and mixed numbers, are written out in words for pupils to write with figures—numerical signs. These dictations *must* use "and" properly.

SUMMARY.

1. The origin of the zero in India. How it reached Europe in form without the abacus. (*A.*)
2. The pupil is taught how the inventor of zero (0) used it to signify "empty"—empty abacus space, how he used it and the other nine numeral signs to indicate "used" and "occupied" spaces, and how in this way he made an abacus unnecessary in expressing written number, completing the present system for integers. (*B, 1.*)
3. Pupils are required to write mixed numbers and fractions on Roman plan (the 10's basis) and read them. This introduced decimal fractions with pupil's improvised "decimal point." (*B, 2.*)
4. The period as the "decimal point," the sign of dividing line between integer and fraction spaces, is taught. (*C, 1.*)
   Exercises are given in reading decimal fractions and in writing fractions from oral dictations. (*C, 2–5.*)
5. The *oral* "decimal point" sign *and.* Pupils are taught that oral *and* is a "decimal point" sign and they are given exercises in its proper use. (*D.*)

Time required for the work of this chapter, including reviews of previous chapters, fifteen days.

CHAPTER V

DECIMAL FRACTION FORMS AND DEFINITION

Topics:—The Denominator Understood. The Denominator Written. Powers of Numbers Developed. The Definition of Decimal Fraction

A. The Two Decimal Fraction Forms

1. Changing from Decimal Form to Common Form

The dictations in this section are written on the blackboard for the pupils to discuss as follows:

3.6. "Read it." ("Three and six tenths.")

"What kind of a number?" ("Mixed.")

"What is the integral part?" ("Three.")

"What is the fraction?" ("6-tenths.")

"What is the numerator?" ("Six.")

"What is the denominator?" ("Ten.")

"Is the denominator written or is it understood?" ("It is understood.")

"How do we understand the denominator?" ("By the position of the 6—in the first fraction space.")

"Write the number with the denominator *written*." ("$3\frac{6}{10}$.")

.21. "Read it."

"What kind of a number?" ("Fractional.")

"What is the numerator?" ("Twenty-one.")

"What is the denominator?"

"Is the denominator written or understood?"

"How do we understand the denominator?"

"Write the number with the denominator written." ("$\frac{21}{100}$.")

Many exercises like the above should be given with the aim to apply in decimal fractions the terms integer, mixed number, and fraction; to use with such fractions the terms numerator, denominator, *denominator written*, and denominator *understood*.

2. *Changing from Common to Decimal Form*

$1\frac{16}{1000}$. "Read it."

"Kind of a number?" ("Mixed.")

"Integral part?"

"The fraction?"

"The numerator?"

"The denominator?"

"Denominator written or understood?" ("Written.")

"Write the number with the denominator understood." ("1.016.")

This is the reverse of *1*. The aim is to complete the establishment of the term "written" and "understood" as applied to denominators. Very many exercises should be given in *2* and then many in *1* and *2* mixed or alternating. In connection with this work in *1* and *2* alternating, the teacher must develop (*B* and *C* below) and then use the definition of decimal fraction. It is very important that this definition be learned and retained.

B. Powers of Numbers

1. The Need of the Term

The proper definition of decimal fraction involves the use of the term power—"whose denominator is some power of 10."

The development of this definition necessitates the previous development of the term *power*. The proper place for the introduction of this term is at this point where the first need of it or opportunity for its use occurs.

2. The Development

The development is naturally in two parts: The written work in which the pupil is able to *see* the factors of each power of a number and the product of these factors. By seeing and counting the factors he more readily understands what 1st power and 2d power and 3d power mean. The oral and mental work.

The following outline indicates the order of work in the development:

a. The written work.

$$\left.\begin{array}{r} 2\times 2\times 2=8\\ 2\times 2=4\\ 2=2 \end{array}\right\}$$ The teacher writes these on the blackboard.

"In the first expression we have 2×2×2. The product is 8."

"We say that 8 is a power of 2—the 3d power."

"In the 2d expression we have 2×2. The product is 4."

"We say that 4 is a power of 2—the 2d power."

"Look at the third expression."

"We say that 2 is the first power of 2."

$$\left.\begin{array}{r} 3\times3\times3=27 \\ 4\times4=16 \end{array}\right\}$$ Written on the blackboard.

"We say that 27 is the 3d power of 3."

"We say that 16 is the 2d power of 4."

"You may write an expression showing the 4th power of 5. The 1st. The 4th."

Other exercises of this kind should be given in which the child is required to write powers of 6, 8, 5, 7, 12, 10, etc. The 3d power with the higher numbers is as high as the exercises need to go.

b. The oral and mental work.

(i) What is the 2d power of 2? (4). The 4th? (16).

What is the 3d power of 3? (27). The 2d? (9).

What is the 2d power of 5? The 1st? The 3d?

The 3d power of 6? The 1st? The 2d?

The 2d power of 10? The 3d? The 1st? The 4th?

The 3d power of 7? Of 5? Of 2? Of 4? Of 3?

The 2d power of 10? Of 8? Of 9? Of 3? Of 5?

(ii) 36 is what power of 6? 16 is what power of 2?

25 is what power of 5? 16 is what power of 4?

81 is what power of 9? 64 is what power of 2?

81 is what power of 3? 64 is what power of 2?

100 is what power of 10? 10 is what power of 10?

1000 is what power of 10? 125 is what power of 5?

(iii) Is 8 a power of 2? (Yes, the 3d).

Is 16 a power of 3? (No.) Is 16 a power of 4? (Yes, the 2d.)

Is 32 a power of 8? Is 48? Is 64? (Yes, the 2d.)

Is 20 a power of 5? Is 5? Is 25? Is 100? Is 125?

(iv) Give me the full series of the powers of 2 to the 4th term. (2, 4, 8, 16.)

Give me the series of the powers of 5 to the 3d term. (5, 25, 125.)

The series of the powers of 6 to 3d term. (6, 36, 216.)

The series of the powers of 10 to the 4th term. (10, 100, 1000, 10000.)

(v) What are 4, 16, 64? (The powers of 4 to the 3d term.)

What are 2, 4, 8, 16, 32? (The powers of 2 to the 5th term.)

What are the following series:

8, 64, 512?
5, 25, 125, 625?
3, 9, 27, 81?
10, 100, 1000, 10000?

(vi) Write the full series of the denominators of a decimal fraction which occupies the first three fraction positions. (10, 100, 1000.)

Write the full series of the denominators of a fraction which occupies the first two positions. (10, 100.)

Write the full series of decimal fraction denominators to the 4th position. (10, 100, 1000, 10000.)

What are these numbers? (The powers of 10 to the 4th term.)

(Exercises like those in (iv), (v), and (vi), are the special preparation for the definition of decimal fraction in *C*, below.

C. Decimal * Fraction Defined

"A decimal fraction is a fraction whose denominator is some power of 10. The denominator may be written or it may be understood." This is the definition. It should be memorized and must be called for very frequently.

$\frac{1}{2}$. "Is it a decimal fraction?" ("No.") "Why?" ("Its denominator is not a power of 10.")

$\frac{3}{4}$. "Is it a decimal fraction?" "Why?"

$\frac{3}{10}$. "Is it a decimal fraction?" ("Yes.") "Why?" ("Its denominator is a power of 10.")

.14. "Is it a decimal fraction?" ("Yes.") "Why?"

Other fractions—$\frac{3}{5}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{2}{10}$, $\frac{3}{20}$, $\frac{4}{50}$, $\frac{25}{100}$, .36, .108, etc.—should be discussed as to whether they are decimal fractions or not.

(*All* fractions having 10, 100, 1000, or 10000, etc., as denominators are decimal fractions whether the denominators are *written* or *understood.*)

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* It would be well in connection with this to teach the definition of *duodecimal* fractions—"A duodecimal fraction is a fraction whose denominator is some power of 12, as 12 or 144 or 1728, etc. Duodecimal fractions were used by the Romans before decimal fractions were invented."

D. Fraction Forms

1. *Forms Described*

Fractions written numerically are in two forms—

(*a*) Common fraction form in which the denominators are written; and,

(*b*) Decimal fraction form in which the denominators are understood.

$\frac{9}{10}$, $\frac{75}{100}$, and $\frac{125}{1000}$ are common fractions because the denominators are written. They are also decimal fractions because their denominators are powers of 10.

.9, .75, and .125 are not common fractions. They are decimal fractions in decimal fraction form.

$\frac{9}{10}$, $\frac{75}{10}$, and $\frac{125}{1000}$ are decimal fractions in common fraction form.

It is necessary, therefore, to use the terms common fraction, common fraction form, and decimal fraction form, and to introduce them here. Their introduction enables the teacher to dictate orally, clearly, and briefly the form in which she may wish a decimal fraction to be written. It enables the pupil to express the fact about $\frac{2}{10}$, that it is a decimal fraction in the common form, and that as .2 it is a decimal fraction in regular form.

The pupil must learn to discriminate between decimal fraction and decimal fraction form. When we speak of decimal fraction we refer to a fraction having as a denominator 10 or 100 or 1000, etc. There is in the expression the presumption that it is to be written in the decimal fraction form. When we speak of decimal fraction *form*, we refer to the method of indicating the denominator—the form in which the position of the numerator from the decimal point determines the denominator.

2. *The Terms Defined*

(*a*) Decimal Fraction.—This is defined in *C*.

(*b*) Common Fraction.—A common fraction is a fraction in which the denominator is written.

(*c*) Common Fraction Form.—The common fraction form is that form in which the denominator is written.

(*d*) Decimal Fraction Form.—The decimal fraction form is that form in which the denominator is understood.

SUMMARY.

1. There are two forms in which decimal fractions may be written—
    1st. The form in which the denominator is indicated by the position of the numerator from the decimal point; and,

2d. The form in which the denominator is written.

Exercises are given in changing decimal fractions from one form to the other. (*A*.)

2. The term power as applied to numbers (the powers of 2, the powers of 5, etc.) is developed preparatory to the development of the definition of decimal fraction. (*B*.)
3. The definition of decimal fraction. (*C*.)
4. The terms "common fraction," "common fraction" form, and decimal fraction form are described. (*D*, *1*.)

Definitions of terms—(*D*, *2*.)
  Common fraction.
  Common fraction form.
  Decimal fraction form.

Time required for the work of this chapter, including reviews, five days.

CHAPTER VI

TRAINING IN READING AND WRITING DECIMAL NUMBERS

Topic:—Emphasis Points in Reading and Writing Decimals: In Fractions—the *Numerator*, the Denominator; In Mixed Numbers—the *Integer*, the *And*, the *Numerator*, the *Denominator*

A. Aim

Very many pupils who may be able to read and write integral numbers with facility, have great trouble in reading and writing decimal mixed numbers that are not perfectly simple. This is not because such numbers are more difficult to read and write than integers but because the individual has not been trained in system and order in such work.

The aims of this section are—

(1) To call the child's special attention to the emphasis points in mixed numbers and fractions (the integer, the and, the numerator, the denominator, in mixed numbers; the numerator, the denominator in fractions) and to the order in which he has been accustomed to read and write common fractions and simple decimal fractions alike (the integer, the and, the numerator, the denominator; or the numerator, the denominator, if the number is a common or a decimal fraction).

When, for illustration, he reads 1.6, he sets off with emphasis and distinct voice pause the following "points"—*one*, *and*, *six*, and *tenths*. The *one* is the integer; the *and*, the notice to himself and to the listener that the reading is passing over from the integral to the fractional spaces of the abacus; the *six*, the numerator; and the *tenths*, the denominator. There is a voice pause after each "point." He reads it, One-and-six-tenths. These are the emphasis points in their proper order. His "points" and order with $\frac{8}{5}$ (a common fraction) are, Eight-fifths; with $8\frac{5}{6}$ (a common fraction mixed number), Eight-and-five-sixths.

(2) To train him to note these emphasis points, to give them invariably in the same order, and to set them off distinctly and emphatically whenever he reads or thinks (for writing) any kind of fractional expression—expecially a decimal fraction. All this is for the purpose of habit-forming in *thinking* fractions properly.

Systematic and orderly reading and writing in decimal fractions consists in thus distinctly setting off and ordering the emphasis points. When the individual acquires the habit of regularly reading and writing decimals with attention to and emphasis upon these "points" in their order as given, an error in reading and writing the difficult expressions will seldom be made. Pupils will not form this habit unless the *teacher* has it.

B. Reading and Writing Fractional Numbers

1. Reading Fractions

The aim in this is to lead the pupil to notice the *order* in which he reads a fraction.

$\frac{3}{4}$. (Written on the blackboard.)

What kind of a number? (A fraction).

Read it. (Three fourths).

What did you read first? (The numerator).

What is the numerator? (Three).

What did you read next? (The denominator).

What is the denominator? (Fourths).

Is this fraction proper or improper? Why?

Other fractions are dictated, read, and questioned upon in the same way—$\frac{5}{8}$, .2, .8, $\frac{2}{3}$, .12, $\frac{12}{15}$, $\frac{21}{32}$, $\frac{19}{20}$, $\frac{8}{7}$, $\frac{12}{5}$, etc.

After two or three of these exercises—$\frac{3}{4}$, $\frac{5}{8}$, and .2—have been read and the questions asked upon them, the full statement of how the individual reads a given fraction should be formulated as follows: "When I read the fraction I read the numerator, then the denominator." This full statement should be required each time after the pupil has read the fraction.

.1 (On the blackboard).

What kind of a number is this? (A fraction).

Read it.

What did you read first?

What is the numerator? (One).

What did you read next?

What is the denominator? (Tenth). Is the fraction proper or improper? Why?

Tell me all about how you read the fraction. (When I read the fraction, I read the numerator, then the denominator.)

2. *Writing Fractions*

Exercises of the same kind are now taken up for writing. The teacher dictates the fraction orally and the pupils write it:

Seven ninths.

What kind of a number is this? (A fraction.)

Write it. ($\frac{7}{9}$.)

What did you write first? (The numerator.)

What is the numerator? (Seven.)

What did you write next? (The denominator.)

What is the denominator? (Ninths.)

Is this fraction proper or improper? Why?

Three tenths.

What kind of a number is this?

Write it with the denominator understood. (.3.)

What did you write first? (The numerator.)

What is the numerator? (3.)

What did you do next? (I put the 3 into the first fraction position.)

What is the denominator? (Tenths.)

Is this fraction proper or improper? Why?

Other fractions—$\frac{5}{8}$, $\frac{2}{10}$,* .04,† $\frac{9}{7}$, $\frac{7}{2}$, .5,† .18,† $\frac{20}{100}$,* .12,† $\frac{12}{7}$, $\frac{9}{8}$ .14,† etc.—are now given in the same way.

After two or three of these have been given, the full statement of how the individual writes a given fraction is formulated, "When I write a fraction, I write the numerator and then the denominator."

With decimal fractions written with the denominator understood the teacher should ask an additional question—How did you write the denominator? (By giving the numerator its position from the decimal point.)

C. Reading and Writing Mixed Numbers

1. *Reading Mixed Numbers*

(*a*) $2\frac{5}{7}$. (On the blackboard).

What kind of a number is this? (A mixed number.)

(* The teacher in dictating directs that it be written with the denominator expressed.)

(† The teacher in dictating directs that it be written with the denominator understood.)

Read it. (Two-and-five-sevenths.)

(The pupil must be required to read this as indicated—with the voice pause.)

What did you read first? (The integer.)

What is the integer? (Two.)

What did you read next? (I read the *and*.)

What does the and mean? (Integers end, fractions begin.)

What did you read next? (The numerator.)

What is the numerator?

What did you read next? (The denominator.)

What is the denominator?

Is this fraction proper or improper? Why?

(*b*) 4.03. (On the blackboard.)

What kind of a number is this? (A mixed number.)

Read it. (Four-and-three-hundredths.)

What did you read first?

What is the integer?

What did you read next? (The and.)

What does the and mean? (Integers end, fractions begin.)

What did you read next? (The numerator.)

What is the numerator?

What did you read next?

What is the denominator?

Is this fraction proper or improper? (Proper.)

(*e*) 3.125. (On the blackboard.)

What kind of a number?

Read it. (Three-and-one hundred twenty five-thousandths.)

What did you read first? etc. Is the fraction proper or improper?

(*d*) Statement Formulated.

After several exercises of this kind—12.21, $4\frac{9}{11}$, $7\frac{32}{35}$, 15.104, 125.0134, 7.0101, 91.9001—the pupils formulate the full statement of how they read a mixed number, "When I read a mixed number, I read the *integer*, then the *and*, then the *numerator*, then the *denominator*."

This full statement must be required with every exercise. It involves the essential points in the system and order that this development seeks to establish.

(*e*) Hereafter, without the questions, the pupil states in full, after reading an exercise (always with the voice pauses), "When I read a mixed number I read first the *integer*, then the *and*, then the *numerator*, then the *denominator*." This full statement will be referred to by the teacher as, The rule for reading mixed numbers. *After* the pupil has stated the rule, it would be well to ask the three questions, What is the numerator? What is the denominator? Is this fraction proper or improper?

Many exercises must be given as follows, each exercise being read with voice attention to the four emphasis points and followed by the rule for reading: 25.125, 306.0016, 28.1011

101.011, 202.0202, 35.202, 70.0701, 38.0707, 700.007, 80.801, 700.01, 200.02, 3000.003, 301.0301, 400 $\frac{4}{10}$, 75$\frac{8}{9}$, 75.0075, 800.00008, etc. The questions as in (*a*) or (*b*) should be used.

2. *Reading Decimal Improper Fractions*

These numbers unless written with the denominator expressed ($\frac{29}{10}$, $\frac{405}{100}$, etc.) are, of course, mixed numbers in form. Each exercise should be read as a mixed number and afterwards as a fraction. The questions or the rule followed by questions should be a part of each exercise.

When the teacher asks, in a mixed number reading exercise, for the numerator, the pupils must give the numerator *as it is written. The pupil should give the numerator of the improper* fraction, only when *such numerator is specifically called for.*

The questions as in *1* are used.

1.1. (Written on the blackboard.)

What kind of a number? (Mixed.)

Read it.

What did you read first, etc.? Is the fraction proper or improper? (Proper.)

Read the number as an improper * fraction. (Eleven-tenths.)

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* Decimals when in decimal form have no improper fractions except as the individual chooses to *consider* a mixed number an improper fraction as here.

What is the rule for reading a fraction? ("When I read a fraction, I read first the numerator, then the denominator.")

What is the numerator? (Eleven.)

What is the denominator? (Tenths.)

Is this fraction proper or improper? (Improper.)

1.303. (On the blackboard.)

What kind of a number?

Read it.

What is the rule for reading a mixed number?

Is the fraction proper or improper? (Proper.)

What is the numerator? The denominator?

Read it as an improper fraction. (One thousand three hundred three—thousandths.)

The rule for reading a fraction?

The numerator? (One thousand three hundred three.)

The denominator?

Is this fraction proper or improper? (Improper.)

Similar Exercises with—

3.05, 5.2, 1.12, 1.101, 80.2, 12.000,* 1.09, 10.101, 4010.1, 5.02, 1.75, 1.09, 1.203, 50.9, 2.51, .1001, 100.1, 401.01, .35, .305, 10.01,

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* Expressions like these should be read as integers—twelve, three, five, six—and afterwards as improper fractions. It would be careless teaching to permit a pupil to read 10.0 as ten and no tenths, or 5.00 as five

12.013, 3.256, 3.00,* .2011, 20.02, 3.007, 4.3, 1.101, 8.11, 61.1, .171, 8.92, 3.001, 40.101, 10.00, 5.00,* .600, 10.201, .01002, 10.02, 5.0, 6.0,* 1020.1, 1.0201, .1002, 100.2, 6.00, 1.06, 102.01, .10201, 1.002, 1002. 8.000.

It is very essential that in reading exercises as improper fractions the pupil set off clearly by voice pause the numerator and the denominator, also that he state the rule for reading a fraction. Systematic reading consists in noting thus very carefully the numerator then the denominator. The question, What is the numerator? is very important and must be asked regularly. Attention is thus brought to the numerator in two ways—by the emphasis in the reading and by the question. The work in this section (2) must be reviewed frequently for months.

3. *Writing Decimal Mixed Numbers and Fractions*

(*a*) Presentation.—It is in this work of writing decimal mixed numbers and fractions, particularly improper fractions, that system and order has practical value for accuracy.

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and no hundredths. 3.00 is *three* as it is written. It is three hundred hundredths as an improper fraction. The pupil should be given considerable experience with such numbers.

If the *integer*, the *and*, the *numerator*, the *denominator*, with mixed numbers; or, the *numerator*, the *denominator*, with fractions improper or proper, has consideration, each in its turn, the resulting written expression will be correct.

The dictations are given at first orally and later in written form, the teacher exercising care to word them correctly—using *and* only between the integral and the fractional parts of a mixed number. 100.1 as an improper fraction is one thousand one tenths; as a mixed number, it is one hundred and one tenth. 1.02, as a mixed number, is one and two hundredths; as an improper fraction it is one hundred two hundredths. 20.09 is twenty and nine hundredths; or, two thousand nine hundredths.

In expressing denominators orally the pupil should be required to say *hundredths*, not one-hundredths; thousandths, not one-thousandths. 1.55 is not one and fifty-five one-hundredths, but one and fifty-five hundredths. The fraction in 3.009 is nine thousandths, not nine one-thousandths.

Three and nine hundredths. (Oral dictation.)

The pupil goes to the blackboard and writes giving orally the emphasis points as he writes—

("This is a mixed number." "The *integer*" (3), "the *and*" (3.), "the numerator" (3.9), "the denominator" (3.09).)

The numbers in the parentheses show the progress of the written work as he talks or states each "point." He writes the 3, then with the "and" the ".", then the 9 as "numerator," then the cipher as establishing the denominator—giving 9 its proper position.

("I wrote the *integer*, then the *and*, then the numerator, then the denominator.")

"How did you write the denominator?" ("By giving the numerator its position from the decimal point.")

One thousand one hundred-thousandths. (Oral dictation.)

("This is a fractional number."

"The numerator (1001), the denominator (.01001)."

"I wrote the numerator, then the denominator.")

Twenty and five hundred thousandths. (Oral dictation.)

("This is a mixed number."

"The *integer* (20), the *and* (20.), the numerator (20.500), the denominator (20.500.)"

"I wrote the integer, then the and, then the numerator, then the denominator.")

(The pupil makes no change when he says

"denominator" because the 5 already has its proper position.)

Two hundred thousand twenty hundredths.

("This is a fractional number."

"The numerator (200020), the denominator (2000.20)."

"I wrote the numerator, then the denominator."

"The number * is two thousand and twenty hundredths.")

Many additional exercises should be given at this time and a few occasionally for weeks later. This rule and the emphasis points should not be neglected. These exercises are for habit forming, hence the need of attention to the way the pupil writes—the integer, then the and, then the numerator, then the denominator—and to his statements of the order of writing (the rule).

NOTE.—The teacher must insist upon a clear-cut adherence to the emphasis points in writing these numbers. In "three and five thousandths," he must write it (talking as he writes) "three" (3)—"and" (3.) —"five" (3.5)— "thousandths" (3.005).

The following illustrate the character of the dictations:

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*Every improper fraction, after it has been written, must be read as a mixed number.

(*b*) Exercises for Oral Dictation.

Three and five thousandths.

Four thousand six hundredths.

One hundred fifty thousand eight tenths.

Seven thousand one hundred nine hundredths.

Two hundred and fifty thousand millionths.

One thousand and two thousandths.

Many others should be given.

(*c*) Exercises in Written Dictation.

These are like those in (*b*) excepting that the dictations are written out in words.

Twenty-two and twenty-two thousandths.

Three hundred and three thousandths.

Forty and forty hundredths.

Two hundred two thousand ten-thousandths

Many others should be given.

In each written-out exercise like this the pupil should state whether the expression is mixed or fractional. If mixed, he should point out the integer, the and, the numerator, the denominator before attempting to write the number. The suggestions in Note under (*b*) must be carefully followed.

(*d*) Exercises for Writing Out the Numbers.

These are exercises dictated in numerical form for pupils to write out the numbers in words.

SUMMARY.

1. The aim in this chapter is to train the pupil to habits of systematic thinking when reading and writing fractions and mixed numbers. There are in reading or writing mixed numbers and fractions four emphasis points—the *integer*, the *and*, the *numerator*, and the *denominator*. (*A*.)
2. Training in reading and writing fractions with statements by the pupil of his *two* emphasis points. (*B*.)
3. Training in reading and writing mixed numbers with statements by the pupil of his *four* emphasis points. (*C*, *1*.)
4. Training in reading decimal improper fractions with statements of the *two* emphasis points. (*C*, *2*.)
5. Training in writing decimal mixed numbers and improper fractions with statements by the pupil of the emphasis points. (*C*, *3*.)

Time required for the work of this chapter with necessary reviews, five days.

CHAPTER VII

DECIMAL FRACTION SERIES AND REDUCTIONS

Topics:—Series of Decimal Fraction Equivalents. Reducing Decimal Fractions to Lower and Higher Terms

A. Aim

1. *Importance of the Work*

The pupil has had training in series work in common fractions and in his computations is accustomed to change fractions to higher or lower equivalents according as necessity or simplicity demands. A fraction in the result of a computation, if not in its lowest terms, would invaribly be reduced to that form. In addition and subtraction he immediately seeks a higher form for each, when necessary, in order that the fractional units may have a common denominator. These changes to lower, lowest, or higher terms have entered so fully into his computation habits that he uses them almost unconsciously. He notes instantly a fractional result that is not in its lowest terms, and computes with fractions only in the lowest form that the operation will permit.

Reductions to higher and lower equivalents have as important a place in decimal as in common fraction computations. For illustration, in the subtraction of .21 from .5, the pupil must change the .5 to its next higher equivalent—.50. In multiplying 36 by 1.50, he should operate with 1.50 in its lowest terms —1.5. In dividing 8.1 by .03, there must be a change in the dividend to 8.10—the next higher term of the series. In finding $\frac{1}{4}$ of 3.1, it is necessary to find $\frac{1}{4}$ of the third term of the 3.1 series—3.100. To tell the pupil that a cipher at the right of the last digit in a decimal fraction may be canceled or that ciphers may be annexed to such a fraction without affecting its value, is arbitrary and makes the operation mechanical. It gives it no basis in reason.

2. *Aim*

The aim in series work in decimals is to develop the principle underlying the canceling or annexing of such ciphers, giving the process in this way a foundation in intelligence.

3. *Kinds of Series*

Fraction series are of two kinds—

1st. Common fraction series in which the terms represent all the possible fraction equivalents; and,

2d. Decimal fraction series in which the terms represent all the possible *decimal* fraction equivalents.

Illustrations. $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, $\frac{15}{20}$, etc. represent, when continued, all the possible equivalents of $\frac{3}{4}$.

.75, .750, .7500, .75000, etc. represent, when continued, all possible decimal fraction equivalents of .75.

B. The Introduction

The step preparatory to work upon decimal fraction series is a few exercises in series from common fractions. Such review gives the pupil the opportunity to make the logical inference, where a series of *decimal* fraction equivalents is suddenly called for, that the problem is merely the extension of a familiar common fraction operation and to write his first series of this kind without being instructed or assisted. The introduction is as follows:

1. The Common Fraction Series Review

$\frac{3}{4}$. (Written on the blackboard.)

Write the series of equivalents to the 4th term.

(The pupils write $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$.)

Questions.

What is the 3d term? The 4th? The lowest?

Are the terms proper fractions or improper?

What is a proper fraction?

What is the numerator of the —— term? Of the —— term? Of the lowest term?

What is the denominator of the —— term? Of the lowest term?

Give the denominators of the series to the 6th term.

Which is the larger the —— term or the —— term? ("They are equivalents.")

Similar work with $\frac{4}{5}$, $\frac{2}{3}$, $\frac{6}{10}$,* $\frac{8}{5}$, $\frac{9}{4}$, $\frac{3}{100}$,* $\frac{9}{10}$,* is now given.

2. *Proper Fraction Series*

a. THE INTRODUCTION

.2 (Written on the blackboard).

"Write the series of decimal fraction equivalents to the 4th term."

The pupils, practically without exception, will write .2, $\frac{4}{20}$, $\frac{6}{30}$.

The teacher's response should be an emphatic repetition of the direction which she gave—

"I told you to write the series of *decimal* fraction equivalents. Listen to me—*Decimal* fraction equivalents.

This will serve to concentrate attention upon the nature of the series to be written and the series .2, .20, .200, .2000 will result.

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* The teacher will understand that the pupils will respond at this time, in such exercises as these, with the common fraction series—$\frac{6}{10}$, $\frac{12}{20}$, $\frac{18}{30}$, $\frac{24}{40}$, etc.; and $\frac{3}{100}$, $\frac{6}{200}$, $\frac{9}{300}$, $\frac{12}{400}$, etc.

b. QUESTIONS

The questions suggested in *1* must be used again here.

NOTE.—The points which the questions suggest are very important in this decimal series work, particularly those which relate to the denominators. The questions on the denominators of the series (10ths, 100ths, 1000ths 10000ths in this case) will bring to the pupil's attention gradually that *all* decimal fraction series have denominators along the single line—10, 100, 1000, 10000, etc. The question about which of two given terms is larger will keep before the pupil the fact of the equality of the terms of a series.

c. KINDS OF EXERCISES

Similar exercises with .6, .8, .12, .25, .231, .101, .1001, etc. The questions should be used with each.

Mixed exercises should follow—the teacher giving an exercise in writing now common fraction equivalents and again decimal fraction equivalents. (A few exercises should be given in writing both a common fraction and a decimal fraction series from one fraction as follows:

.4, .18, $\frac{6}{100}$, $\frac{9}{10}$, $\frac{12}{1000}$, .11, .15, etc. In writing the decimal equivalents of $\frac{6}{100}$, $\frac{9}{10}$, etc., the pupil should be required to preserve the common fraction form and write the common fraction series—$\frac{6}{100}$, $\frac{60}{1000}$, $\frac{600}{10000}$, etc.; $\frac{9}{10}$, $\frac{90}{100}$, $\frac{900}{1000}$, etc. The decimal fraction equivalents should be written when the dictation is in regular decimal fraction form—.8, .55, .102, etc.—and the common fraction equivalents when the dictation is in common fraction form, unless the teacher should directly ask for decimal fraction equivalents.)

These exercises fix the fact of two distinct classes of series of equivalents and in what respect they differ. The questions should be used with each exercise.

3. *Mixed Number and Improper Fraction Series*

1.01 (Written on the blackboard.)

What kind of a number? (Mixed.)

Read it.

What is the numerator? The denominator?

Read it as a fraction. (One hundred one hundredths.)

What is the numerator? The denominator?

Write the series of equivalents to the 3d term. (1.01, 1.010, 1.0100.)

What are the denominators of the series to the 4th term?

Read the 3d term. Read it as a fraction. (Ten thousand one hundred ten-thousandths.)

Read the lowest term. Read it as a fraction.

Which is the greater, the —— term or the —— term? (They are equivalents.)

Similar exercises with 3,002, 15.001, 8.2, 18.11, 25.202, 5.01, 10.001, 3.0101, 11.011, etc.

C. Finding the Number of the Term and the Lowest Term

.70 (Written on the blackboard.)

This is one of the terms of a series of decimal fraction equivalents. Which term is it? (The 2d term.)

What would the 4th term be? (.7000.)

The lowest term? (.7.)

Read the term. The lowest term.

Which is greater the .7000 or the .70? (They are equivalents.)

What are the denominators to the 5th term?

What is the numerator of the 3d term? Of the 5th term?

8.500. (Written on the blackboard.)

This is a term of a series of decimal fraction equivalents.

Which term is it? (The 3d.)

Write the 5th. The lowest. The 2d.

Read the 5th. Read it as a fraction. (Eight hundred fifty thousand hundred-thousandths.)

The numerator of the 4th term? The numerator of the 4th term as a fraction?

What are the denominators to the 5th term? (10, 100, 1000, 10000, 100000.)

The denominators, as a fraction, to 5th term? (10, 100, 1000, 10000, 100000.)

Which is greater the 8.500 or the 8.5?

Which is greater the 8.50 or the 8.50000?

Read the lowest term. The 5th term.

Many other similar exercises with 10.10, 1.01, .010, 1.011, 2.303, 1.5, 3.00,* 5.00,* 3.010, .42, .015, 4.00,* 6.0,* .3, .40, .60, .01050, 25.00,* 7.0,* 1.100, etc. There should be dozens of such exercises and the step should have frequent reviews for weeks.

D. Reductions to Lower and to Higher Terms

This completes the development of decimal fraction equivalents. Hereafter, in all computations or exercises in which it may be necessary to annex a cipher to a number, it must be spoken of as reducing to higher terms. When the result of an exercise or problem is not in its lowest terms it must not be accepted. The removal of a cipher or ciphers in order to effect the proper reduction must be spoken of as reducing to lowest terms. The teacher must be as consistent in requiring decimal fraction results to be in lowest terms as in computations with common fractions. (See Chapter XII, *B*, *3*.)

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* The pupils should regard expressions like these as integers. He will read them as integers—three, five, four, six, etc. When asked to read them as improper fractions, he will read them—three hundred hundredths, five hundred hundredths, sixty tenths, etc.

SUMMARY.

1. The importance of series changes—changes to higher terms and to lower terms—in the fundamental operations where decimal fractions are involved—
    (1) In making additions, subtractions, and divisions possible in many computations; and,
    (2) In simplifying or shortening operations. (*A*, *1*.)
2. Common fraction equivalent series reviewed as the basis for the development of series from decimal fractions in decimal fraction form. (*B*, *1*.)
3. Series of decimal fraction equivalents with proper fractions developed by requiring the pupil to adhere to the language of the direction given them "*decimal fraction equivalents*." In this way the pupil learns to discriminate between fraction equivalents and decimal fraction equivalents. (*B*, *2*.)

4. Decimal fraction equivalents from mixed numbers and improper fractions. (*B*,*3*.)
5. Finding the number of the term in the series and the lowest term when a single term of a decimal fraction series is given. (*C*.)
6. Reductions to Lower and to Higher Terms.

   The pupil is taught that all results of operations must be in lowest terms and that annexing ciphers or removing ciphers must be spoken of as reductions. (*D*.)

Time required for the work of the chapter, including necessary reviews, fifteen days.

CHAPTER VIII

ADDITION, SUBTRACTION, AND MULTIPLICATION

Topic:—The Development of "Pointing Off" in addition, Subtraction, and Multiplication

A. Addition

1. Written Dictations

There need be no explanations or instructions in this work. The child added on the abacus and that work is the only preparation required. Dictations should be given in horizontal form, the child being required to write and add the exercises vertically:—

25.1 + 2.2
3·6 + 3·4
3·5 + 2.6
4·93 + 3·08

.024 + 2.8
.003 + .02
24 + .005
1.67 + 24.019

.001 + 3.
2.67 + 1.009
·39 + 1.005
·46 + .087

The child will learn by experience that there is greater convenience in adding by writing units of the same kind under each other. This arrangement of "units under units and tenths under tenths," as it is sometimes loosely expressed, is secured by conducting the work for the first dozen or more exercises as class work—one child, then another, and then another going to the blackboard to do the work and giving with it the *full* oral work.

In $\begin{array}{r} 25.1 \\ +2.2 \\ \hline \end{array}$ the pupil writes the expressions in vertical form and then begins to "talk." ("Two tenths and one tenth are three tenths, two and five are seven, I write two tens. The sum is twenty-seven and three tenths.")

In $\begin{array}{r} 4.93 \\ +3.08 \\ \hline \end{array}$ the pupil at the blackboard says, "8 hundredths and 3 hundredths are 11 hundredths. This is 1 tenth and 1 hundredth. I write the 1 hundredth. 1 tenth and 9 tenths are 10 tenths. This is 1. I write tenths space empty. 1 and 3 and 4 are 8. I write 8. The sum is 8 and 1 hundredth."

The teacher will observe that the full oral work will lead the child to discover the awkwardness of adding when one of the addends is in one column and the other or the others in another column, and to arrange units of the same name under each other for convenience. If in .024 2.8 the pupil should write $\begin{array}{l} .024 \\ +2.8 \\ \hline \end{array}$ his difficulties in adding 8 tenths and 4 thousandths (if he should attempt so to add) will lead him to correct himself as to adding and as to arrangement, *if only the teacher will wait and let him fight it out alone.*

Delay in order to give the child a chance to overcome the difficulty alone is time well spent.

2. *Addition—Oral Dictation*

These exercises are like those in the section above excepting that the dictation is oral. It gives additional experience in writing decimals, as well as in addition. Exercises should be given in blackboard solutions with full oral work as in the section above.

B. Subtraction

1. *Written Dictation*

No explanation or instructions should be given. The dictations should be written on the blackboard in horizontal form—3.6 − 1.4 =, 14.009 − 2.12 =, etc.—and the pupils required to write and subtract in the usual vertical form. This work as in addition should be introduced by blackboard exercises with full oral work as there outlined.

In $\begin{array}{r} 3.84 \\ -1.25 \\ \hline \end{array}$ the oral work by the pupil who is at the blackboard is "5 hundredths from 4 hundredths I cannot take. I take 1 tenth from the 8 tenths leaving 7 tenths. The 1 tenth is equivalent to 10 hundredths. I have now 14 hundredths. 5 hundredths from 14 hundredths leave 9 hundredths. 2 tenths from

7 tenths leave 5 tenths. 1 from 3 leaves 2. The remainder is 2 and 59 hundredths." (If required to read it the "long way," the child will read 2 and 5 tenths and 9 hundredths.)

2. *Oral Dictations*

These are given as much for the additional experience in writing numbers as for that in subtraction.

C. Multiplication

1. *"Pointing Off"*

This work should begin by simple exercises of tenths multiplied by tenths, hundredths by tenths, tenths by hundredths, etc.—the aim being to give the child an opportunity to find out for himself how to "point off" the products. This (the pointing off) is determined by the pupil by performing the operation in each case in *common fraction form.*

For illustration, $.5 \times .7$ is arranged by the pupil in the usual multiplication form—$\left(\begin{array}{r} .5 \\ \times .7 \end{array}\right)$.

In solving, the child must "talk," giving the *full* oral work. "7 tenths times 5 tenths is —" (He will almost surely say 35 tenths). "Prove it." If the pupil hesitates, not knowing how to make a "proof," the teacher may make the suggestion, "We prove by solving it the *old* way." This the child will understand. He is to write the expression as $\frac{7}{10} \times \frac{5}{10} =$ and solve in that form. $\frac{7}{10} \times \frac{5}{10} = \frac{35}{100}$. Therefore

$$\begin{array}{r} \cdot 5 \\ \times \cdot 7 \\ \hline \cdot 35 \end{array}$$

"Pointing off" is developed by requiring this *proof* in *every* case for the first few exercises whether the child points off correctly or incorrectly. Another and then another exercise in tenths by tenths is given and proof required in each case. The results of the proofs must be put into *oral* form and given at the conclusion of each proof—"Tenths times tenths give hundredths." This rule is not to be *given* to the child. It must be a statement of what his proof revealed.

The class work in this development of pointing off must be in the form of class exercises in which one child and then another goes to the board with an exercise and "talks" as he writes.

After two or three tenths by tenths exercises as above, there should be a change to exercises like .06 × .3. The child may try to arrange it in the same form as addition and subtraction exercises—tenths under tenths, etc.

$$\begin{array}{r} .06 \\ \times .3 \\ \hline \end{array}$$

The teacher may tell him, if she chooses that such arrangement in multiplication is not necessary.

He may write this exercise

$$\begin{array}{r} .06 \\ \underline{\times .3} \end{array}$$

The oral work is "3 tenths times 6 hundredths is ——." The child will say either that the product is 18 tenths or 18 hundredths. The teacher must in *any* case require *proof*. This proof is, of course, $\frac{3}{10} \times \frac{6}{100} = \frac{18}{1000}$. The oral statement of what the proof reveals is "Tenths times hundredths give thousandths."

Other exercises under this head:

$$\begin{array}{r} .8 \\ \underline{\times .06} \end{array} \qquad \begin{array}{r} .9 \\ \underline{\times .04} \end{array} \qquad \begin{array}{r} .09 \\ \underline{\times .5} \end{array} \qquad \begin{array}{r} .07 \\ \underline{\times .03} \end{array} \qquad \begin{array}{r} .08 \\ \underline{\times .09} \end{array}$$

The next exercises should be in the form of .3 × .18, .16 × .7, .14 × .8, .3 × .13, etc.

The form of work is the same—the pupil pointing off each partial product and proving the work. In .16 × .7 =, the pupil arranges the numbers in the vertical form

$$\begin{array}{r} .16 \\ \underline{\times .7} \end{array}$$

("7 tenths times 6 hundredths is 42 thousandths.")

(Proof—$\frac{7}{10} \times \frac{6}{100} = \frac{42}{1000}$, "Tenths times hundredths gives thousandths.")

The pupil writes the partial product as in any ordinary multiplication and *points it off*. This first partial product will be written—

$$\begin{array}{r} .16 \\ \underline{\times .7} \\ .042 \end{array}$$

("7 tenths times 1 tenth is 7 hundredths." (Proof). "Tenths times tenths gives hundredths.") The written work is now in the form—

$$\begin{array}{r} .16 \\ \underline{.7} \\ .042 \\ \underline{.07} \\ .112 \end{array}$$

"How many partial products?" ("Two.")

"Which has the highest fraction?" ("The first, 2 thousandths.")

"What is the highest fraction in the product?" ("2 thousandths.")

The partial products are now added and the complete product read by the pupil in "long way" and in "short way." The teacher will note that in this work *each partial product must be pointed off.* Otherwise the multiplication, the arrangement of the partial products, and the addition of the partial products is exactly like multiplication of integral numbers.

2. *Fraction Terms and Definitions Reviewed*

The terms numerator, denominator, integral number, fractional number, and mixed number must be brought into use by frequent questions on the numbers in use.

.13 × .24.

"In the .13 what is the numerator?"

"The denominator?" "Is the denominator written?" ("No, it is understood.")

"What kind of a number?" (Fractional.")

"In the .24 what kind of a number?" ("Fractional.")

"What is the numerator?"

"The denominator?"

"Is the denominator written?"

"Express it with the denominator written." ("$\frac{24}{100}$.")

$$\begin{array}{r} .24 \\ \times.13 \\ \hline .0012 \\ .006 \\ .004 \\ .02 \\ \hline .0312 \end{array}$$

"How many partial products?" ("Four.")

"Which has the highest fraction?" ("The first, the .0002.")

"What is the highest fraction in the product?" (".0002.")

3. *Numbers to be Used in Development of "Pointing Off"*

The aim in the class exercises in "pointing off," in which the pupil goes to the blackboard and "talks" and points off each partial product as above, is to give only simple multiplications—those in which there are no "talking" difficulties. The following are illustrative of the exercises referred to:

.35 × 8.2	3.005 × 2.002	.582 × .026
6.1 × .35	7.013 × 8.201	3.821 × .456
2.84 × 3.12	5.607 × 2.05	2.13 × 3.12
2.921 × 4.106	.396 × .284	.015 × .023

Many exercises of this kind should be given for class exercises and many more for seat written work. The questions as to the number of partial products, the highest fraction, etc. should be used.

The teacher will observe that the exercises are fractional and mixed, that the fraction or the fractional part in no number is higher than thousandths, and that no integral part is higher than units of the first order. To multiply 32.3 by .4 would involve "4 tenths times thirty" in the language (the "talking") part. This language is difficult and there is no compensating advantage from its use. Exercises of that kind belong to the "short way" work later on.

4. *Anticipating the Denominator in the Product*

The aim of this section is to develop the power to determine the denominator of the complete product before the multiplication is begun. In $\begin{array}{r} .33 \\ \times .45 \\ \hline \end{array}$ the child is to decide before he begins to multiply that the highest fraction in the product (reading the "long way") will be ten thousandths. This being stated, the child proceeds to multiply.

(*a*) Preliminary to this, it will be necessary to give a few exercises in reading—

3.167.

"Read this the long way."

"What is the highest denominator?" ("Thousandths.")

"The highest fraction?" ("Seven thousandths.")

"The lowest denominator?" ("Tenths.")

.23 × .35.

$$\begin{array}{l} .23 \\ \times .35 \\ \hline .0015 \\ .010 \\ .009 \\ .06 \\ \hline .0805 \end{array}$$

"What is the highest denominator in .23?" ("Hundredths.")

"The highest fraction?" ("Three hundredths.")

"The highest denominator in .35?"

"The highest fraction?"

"The highest denominator in the .0015?" ("Ten thousandths.")

"The highest denominator in the product?"

"The highest fraction?"

After a few exercises in this preliminary reading work, the child is ready to learn to anticipate the highest denominator in products.

(*b*) How to Anticipate the Highest Denominator.

The child has had several exercises in pointing out the highest denominator *after* the multiplication. We now put before him (on the blackboard in a class exercise) an exercise—

$$\begin{array}{r} 2.4 \\ \times .2 \\ \hline \end{array}$$

"What will the highest denominator be?" ("Hundredths.")

"What will the fraction be?" ("8 hundredths.")

If the child is unable to give the answer to the first question, the teacher should not help or suggest. She should have him proceed with the multiplication and answer the above questions *afterwards*. He is now given another exercise $\begin{array}{r} .26 \\ \times .4 \\ \hline \end{array}$ and the question, "What will the highest denominator be?" is asked again. It is rare that the child fails to answer the question with the second exercise. If, however, he fails to answer, he must multiply (giving the oral work and pointing off the partial product) and answer the question after the work is completed.

If he answers the question, he proceeds with the oral work and the pointing off as in all previous work.

The whole aim in this work is to have him state *in advance* what the highest *denominator* will be and what the highest *fraction*.

Having stated these he proceeds with the pointing off of each partial product and of the complete product. Several exercises of this kind should be given before taking up the next section. The pupil must be required to read the complete product of each exercise "the long way" and "the short way." He should be questioned on each complete product—

What is the lowest fraction, long way?
What is the highest fraction, long way?
What is the denominator, short way?
What is the fraction, short way?

These exercises with the questions on each will develop power to "point off the short way"—the next step. The aim is not to *tell* or *show* the child how to point off the complete product in the next step. His habit of thinking, as the result of these exercises, will be to find out the "highest fraction" and "the denominator the short way," and then to place the period—the "stopping place" mark—so as to give that denominator.

5. *Multiplying and Pointing off the "Short Way"*

(*a*) The form of work in this step—class exercises.

After the pupil has become sure in anticipating the highest denominator and fraction as in the previous section, he is ready to take up the work of this section, *but not before.*

(*b*) The aim of this step is to teach the child to perform the multiplication without regard to the fractions, to multiply as if the multiplier and multiplicand were integers, and then to insert the decimal point so as to give the fraction its proper denominator.

$$\begin{array}{r} .34 \\ \times .26 \\ \hline \end{array}$$

The child must state *in advance* what the highest denominator is (ten thousandths) and what the fraction (4 ten-thousandths). He then multiplies as if the numbers were integers—

$$\begin{array}{r} .34 \\ .26 \\ \hline 204 \\ 68 \\ \hline 884 \end{array}$$

He now makes the "4" ten-thousandths, .0884.

(*c*) The questions are now asked on the complete product—

What is the product, long way?
The lowest fraction, long way?
The highest fraction, long way?
The denominator, short way?
The fraction, short way?

The form of work in this step is by class exercises in which one pupil and then another goes to the blackboard to "talk" and write. *Very many* exercises must be given. The teacher will realize that this step is a very important step in multiplication. It establishes the power to "point off." In each of the exercises the child should be required to state *in advance* what the highest denominator in the complete product will be.

6. *Seat Exercises in Multiplication "Long Way"*

Many exercises should now be given in seat work in multiplying. In these exercises the pupil performs the multiplication on his paper, pointing off the partial and the complete products.

7. *Seat Exercises in Multiplying "Short Way"*

Very many exercises in multiplying *without* pointing off the partial products, the child to point off only the *complete* product—

3.26 ×.015	2.901 ×34.2	6.501 ×.0023	.0215 ×31.6	284 ×.04	.0034 × 28
211.6 ×23.4	.0004 ×.0002	2.814 ×.0012	45.6 ×2.04	2815 ×.024	.001 ×.006

SUMMARY.

1. Exercises in addition and subtraction in which the pupil learns by experience to write the numbers in order ("units under units and tens under tens") for the operations and to place the decimal point in its proper place in the results. (*A*.) and (*B*.)

2. Exercises in multiplication in which the pupil is required to prove the denominator of each partial product by performing the operation in common fraction form and to "point off" accordingly. In these exercises the denominator of the product is ascertained *after* the operation. (*C*, *1–3*.)
3. Exercises to give opportunity to determine by a single partial product what the denominator of the smallest fraction and of the fraction will be in advance. This completes the development. (*C 4–7*.)

Time required for this work, twenty days.

CHAPTER IX

REDUCTION OF COMMON FRACTIONS TO DECIMALS

Topic:—The Analysis of the Reduction Process

A. Oral Reductions

$\frac{1}{2}$. "Is this a decimal fraction?" ("No.") "Why?" ("Because the denominator is not a power of 10.")

"May it be changed to a decimal?" ("Yes, into $\frac{5}{10}$.")

"Write it in its lowest terms." (".5.")

"Write the fourth term of the decimal fraction equivalents." (".5000.")

"Read it."

"Write it as thousandths." (".500.")

"Read it."

$\frac{1}{5}$, $\frac{1}{4}$, $\frac{3}{5}$, $\frac{3}{4}$, $\frac{1}{20}$, $\frac{1}{40}$, $\frac{1}{50}$, $\frac{3}{50}$, $\frac{3}{20}$, $\frac{1}{25}$, $\frac{3}{25}$, are now taken up in this same manner.

The child's mental process in all such reductions will be somewhat as follows, using $\frac{1}{4}$ as an example, the pupil asking and answering the questions:

("Can I make it into 10ths?" "No." "Can I make it into 100ths?" "Yes." "$\frac{1}{4}$ is the same as 25 100ths.") He will then write it $\frac{1}{4} = .25$.

If $\frac{1}{8}$ or $\frac{1}{40}$ is given, he will think 10ths, then

100ths, then 1000ths. In finding the equivalent of $\frac{1}{40}$ in 1000ths he should be permitted to use paper and pencil. Most of the questions here should be oral but involving only simple reductions. Much of this should be done for the sake of the mental work.

B. Written Reductions—Even

The aim of this step is to develop the form of the written analysis. This should be required for several weeks in all reductions of common fractions to the decimal form. In order not to complicate the work while developing this form, exercises should be selected that will reduce to *simple* * decimals.

1. The Significance of the Fraction

$\frac{1}{2}$. "What does it mean?" ("It means $\frac{1}{2}$ of 1 — $\frac{1}{2}$ of a whole.")

$\frac{4}{5}$. "Meaning?" ("$\frac{1}{5}$ of 4.")

In this way review the partition equivalents of common fractions—$\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{5}{7}$, $\frac{1}{20}$, $\frac{4}{12}$, etc. (See Chapter XXVI (*J*) of Part I.) The child must *write* the statement in response to oral and written dictations—

$\frac{1}{2} = \frac{1}{2}$ of 1, $\frac{2}{3} = \frac{1}{3}$ of 2, $\frac{1}{20} = \frac{1}{20}$ of 1, etc.

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* Decimal like common fractions may be *simple* or *complex*. A complex decimal fraction is one whose numerator is a fraction or a mixed number. $.3\frac{1}{3}$, $.201\frac{1}{7}$, and $.004\frac{1}{5}$ are complex decimal fractions.

2. *The Written Analysis*

The fractions to be used in this step are the same as those used for the Oral Reductions above—fractions that may be changed evenly to 10ths, 100ths, or 1000ths, etc. Many of the fractions used here may be so simple as to be easily changed to decimal form mentally. They are as suitable as any for this written work because the aim here is the *analysis form.*

a. CHOOSING THE DENOMINATOR

The child's first act in changing a common fraction to decimal form is invariably to decide, mentally if possible, the denominator.

$\frac{1}{2}$. ("I can make it into 10ths.")

$\frac{1}{8}$. ("I cannot make it into 10ths or 100ths. I can make it into 1000ths.")

$\frac{1}{20}$. ("I cannot make it into 10ths. I can make it into 100ths.")

b. THE ANALYSIS

Having decided upon the denominator as above, the child begins his written analysis.

This analysis must be taught. It hardly admits of development with pupils of this age.

"Change $\frac{1}{2}$ to decimal form."

"(I can make $\frac{1}{2}$ into 10ths.")

($\frac{1}{2} = \frac{1}{2}$ of 1.

$1 = \frac{10}{10}$.

$\frac{1}{2}$ of $1 = \frac{1}{2}$ of $\frac{10}{10} = \frac{5}{10} = .5$.)

"$\frac{1}{8}$ to decimal form."

("I can make 8ths into 1000ths.")

($\frac{1}{8} = \frac{1}{8}$ of 1.

$1 = \frac{1000}{1000}$.

$\frac{1}{8}$ of $1 = \frac{1}{8}$[*] of $\frac{1000}{1000} = \frac{125}{1000} = .125$.)

"$\frac{7}{20}$ to decimal form."

("I can make 20ths into 100ths.")

($\frac{7}{20} = \frac{1}{20}$ of 7.

$7 = \frac{700}{100}$.

$\frac{1}{20}$ of $7 = \frac{1}{20}$ of [†] $\frac{700}{100} = \frac{35}{100} = .35$.)

"$\frac{9}{25}$ to decimal form."

("I can make 25ths into 100ths.")

($\frac{9}{25} = \frac{1}{25}$ of 9.

$9 = \frac{900}{100}$.

$\frac{1}{25}$ of $9 = \frac{1}{25}$ of [‡] $\frac{900}{100} = \frac{36}{100} = .36$.)

NOTE.—The reason for the "actual division" in the suggestions above is that by actual division we prepare for the rule for reducing common fractions to decimal form. The teacher will observe that in the statement in each case above there is in effect the "annexing of ciphers to the numerator" of the original fraction "and dividing (see footnotes) by the denominator."

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[*] The pupil should be encouraged to find $\frac{1}{8}$ of $\frac{1000}{1000}$ by actually dividing

$$\begin{array}{r} .125 \\ 8\overline{)1.000} \end{array}$$

[†] The pupil should actually divide

$$\begin{array}{r} .35 \\ 20\overline{)7.00} \end{array}$$

[‡] Actual division

$$\begin{array}{r} .36 \\ 25\overline{)9.00} \end{array}$$

In there is—

$\frac{3}{5}=\frac{1}{5}$ of 3.
$3=\frac{30}{10}$.
$\frac{1}{5}$ of $3=\frac{1}{5}$ of $\frac{30}{10}=\frac{6}{10}=.6$.

Now $\frac{3}{5}=\frac{1}{5}$ of $\frac{30}{10}=\frac{6}{10}=.6$. The first statement ($\frac{3}{5}$) s the fraction to be reduced. In the second statement, $\frac{1}{5}$ of $\frac{30}{10}=\frac{6}{10}$, we in effect "annexed a cipher to the numerator" of the $\frac{3}{5}$ and "divided by its denominator." This analysis leads to that old rule.

Many exercises must be given in even work until the child can follow the analysis form.

C. Written Reductions—Miscellaneous

This step has to do with the reduction to decimal form of any fraction whether it may be reduced evenly or not. It introduces the child to complex decimals—a term that he must learn and use.

The fact that there are some common fractions that may not be changed evenly to decimal form should be brought to the child by two or three *oral* exercises as follows:

"Change $\frac{1}{3}$ to decimal form."

(In response to this the pupil tries (mentally) 10ths, then 100ths, then 1000ths. He replies that it can not be changed evenly.)

"Change $\frac{1}{6}$ to decimal form."

(The child will try this with the same result.)

"Change $\frac{1}{12}$ to decimal form."

(This in the same manner will prove impossible.)

D. Reducing to Definite Denominator

In dictating fractions for reduction to decimal form, the teacher should now state in each case the denominator to which the reduction is to be made. For illustration, the child is told to reduce $\frac{1}{3}$ to 10ths, to reduce $\frac{1}{3}$ to 1000ths, to reduce $\frac{5}{6}$ to 100ths, etc. The child in response reduces to the denominator suggested using the full analysis *written form*, as in *even* reductions above, and writes the *exact* decimal equivalent—the complex decimal fraction as follows:

$\frac{2}{3}$ to decimal form—to 1000ths.
$\frac{2}{3} = \frac{1}{3}$ of 2.
$2 = \frac{2000}{1000}$.
$\frac{1}{3}$ of $2 = \frac{1}{3}$* of $\frac{2000}{1000} = 666\frac{\frac{2}{3}}{1000} = .666\frac{2}{3}$.

Change $\frac{4}{7}$ to decimal form—100ths.
$\frac{4}{7} = \frac{1}{7}$ of 4.
$4 = \frac{400}{100}$.
$\frac{1}{7}$ of $4 = \frac{1}{7}$* of $\frac{400}{100} = 57\frac{\frac{1}{7}}{100} = .57\frac{1}{7}$.

$\frac{7}{12}$ to 10000ths—decimal form.
$\frac{7}{12} = \frac{1}{12}$ of 7.
$7 = \frac{70000}{10000}$.
$\frac{1}{12}$ of $7 = \frac{1}{12}$* of $\frac{70000}{10000} = 5833\frac{\frac{1}{3}}{10000} = .5833\frac{1}{3}$.

It is almost needless to say that the child should have very many exercises in these reductions and that the term *complex decimal fraction* should be in constant use.

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* The teacher should require the actual written division in these cases $\begin{array}{r} .666\frac{2}{3} \\ 3\overline{)2.000} \end{array}$ $\begin{array}{r} .57\frac{1}{7} \\ 7\overline{)4.00} \end{array}$ $\begin{array}{r} .5833\frac{1}{3} \\ 12\overline{)7.0000} \end{array}$

Complex Decimal Fraction Defined. A complex decimal fraction is one whose numerator is a mixed number or a fraction.

(A *simple* decimal fraction is a fraction whose numerator is *not* a mixed number or a fraction.)

E. Forms of Reductions

1. The Long Form

There are two "forms" of reductions—the *long* form (the *analysis* form as above) and the *short* form. The latter is a close approach to our ordinary way of reducing common fractions to decimal form. This *short form* must be taught the children and hereafter, in dictating exercises in such reductions, the teacher must indicate with each exercise whether the reduction shall be performed in the *long* or in the *short* form.

2. The Short Form

This form practically leaves out the first two analysis steps, using only an abridgment of the third.

To reduce $\frac{5}{9}$ to decimal form—to 1000ths:—

$\frac{5}{9} = \frac{1}{9}$ of $555\frac{5000}{1000} = \frac{\frac{5}{9}}{10000} = .555\frac{5}{9}$.

After a few exercises in this form there should be a still further shortening—

$\frac{5}{9} = \frac{1}{9}$ of $\frac{5000}{1000} = .555\frac{5}{9}$.

This latter form is the "short form" that the pupil should use exclusively after the first few exercises, except for an exercise now and then to keep the analysis form (the long form) fresh. He must not be permitted to lose the long form. In all these exercises the *actual division* should be performed as suggested above.

SUMMARY.

1. Exercises in mental work in changing simple fractions to decimal equivalents. This is training in finding the denominators of the decimal fractions and in stating the mental processes in the reductions. (*A*.)
2. Exercises in written reductions, using common fractions that may be reduced to *simple* decimals—even reductions. The child having been taught the significance of fractions, uses the analysis form that he used in his mental reductions in *A*. (*B*.)
3. Exercises in reducing fractions to definite denominations regardless of whether the decimal fraction equivalent is simple or complex. (*C* and *D*.)
   Long form reduction processes. (*E*, *1*.)
   Short form reduction processes. (*E*, *2*.)

Time required for this work, including reviews of previous work, fifteen days.

CHAPTER X

COMPLEX DECIMAL FRACTIONS

Topics:—Reductions of Simple and Complex Decimals to Lower Complex Forms. Reductions of Complex Decimals to Higher Forms, Simple or Complex

A. Aim

In reducing common fractions to decimal form, the child found and wrote decimal fractions that were complex. In those exercises the numerators of the fractions were mixed numbers. The aim of this step is to teach him—

(1) To read and write decimal fractions whose numerators are fractions—.0$\frac{1}{3}$, .000$\frac{3}{4}$, .00$\frac{2}{5}$, etc.; and,

(2) That a common fraction—$\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{5}$, etc.—can not and must not be used alone as a position sign. (This means that a fraction position—first fraction space, fourth fraction space, second fraction space, etc.—may be indicated *only* by a numeral or by the sign "empty" (0); that a fraction alone—not at the right of a numeral or of an "empty" sign—can not be made to occupy a fraction space by placing a decimal point before it.)

(3) To reduce complex decimals to higher and lower terms.

B. The Introduction

1. Reading Complex Fractions—Written Dictations

$.0\frac{1}{2}$. "Read it."
("One-half of a tenth").
"How many fractional positions used?"
("One").
"How many occupied?"
("One—occupied by a *fraction* of a tenth, $\frac{1}{2}$").

$.00\frac{2}{3}$. "Read it."
("Two-thirds of a hundred").
"How many fractional positions used?"
("Two").
"How many occupied?"
("One, the second—occupied by a fraction of a hundredth, $\frac{2}{3}$").

$.2\frac{1}{4}$. "Read it."
"How many spaces used?" "Occupied?"
"In what space is the $\frac{1}{4}$?"
("In the first").

$.000\frac{3}{5}$. The pupil reads it and answers the questions as to used and occupied spaces.
"In what space is the $\frac{3}{5}$?"
("In the third").

Several additional exercises of the same kind should follow— $.12\frac{1}{3}$, $.2\frac{2}{3}$, $.3\frac{1}{4}$, $2.4\frac{2}{7}$, $14\frac{1}{5}$, $.31\frac{3}{4}$, $.2\frac{1}{2}$, $2\frac{1}{2}$, $5\frac{2}{3}$, etc.

The teacher will note that in $5\frac{1}{3}$ the $\frac{1}{3}$ is in the first integral position.

2. Writing Fractions—The Positions of Fractions

"Write $\frac{3}{4}$ in the first fractional position." ("$.0\frac{3}{4}$").

"Read it." ("Three fourths of a tenth").

"Write $\frac{2}{3}$ in the 3d fractional position." ("$.000\frac{2}{3}$").

"$\frac{5}{6}$ in first integral position." ("$\frac{5}{6}$").

"Read it." ("Five-sixths").

"$\frac{5}{7}$ in 2d fractional position." ("$.00\frac{5}{7}$").

"$\frac{3}{5}$ in the 4th fractional position." ($.0000\frac{3}{5}$).

Several additional exercises of the same kind.

3. Writing Fractions—"Position" Applied

"Write $\frac{2}{3}$ of a thousandth." ($.000\frac{2}{3}$).

"$\frac{3}{5}$ of a tenth." "$\frac{5}{6}$ of a ten-thousandth." "$\frac{5}{9}$ of a hundredth."

Many other similar dictations.

NOTE.—The teacher may find it necessary in this step to anticipate *C* below in so far as to teach that a decimal point before a common fraction alone has no meaning.

C. The Decimal Point Not a "Common Fraction" Sign

AIM.—In this step the teacher writes all sorts of possible and impossible complex decimal fractions for the pupil to read or to declare impossible. It is the final test of the pupil's knowledge of decimal fraction written language. This, in effect, reviews what has been taught in *B*.

The aim is to teach (tell) the pupil that the decimal point *can not be used* with a fraction alone—that the decimal point in $\cdot\frac{1}{3}$ has no significance, the $\frac{1}{3}$ in such case not being affected by the "·" the expression $\frac{1}{3}$ and $\cdot\frac{1}{3}$ each meaning one-third of a whole.

This is a *very* important matter—*one of the most important in decimal fraction teaching*—and must be handled forcefully and the lesson driven deep.

Dictations Written—Pupils Read.

$.0\frac{3}{4}$, $.01\frac{2}{3}$, $.00\frac{3}{4}$, $\frac{3}{4}$, $\cdot\frac{2}{3}$, $\cdot\frac{5}{9}$, $.0\frac{5}{7}$, $\cdot\frac{5}{7}$, $\cdot\frac{3}{8}$, $.0\frac{7}{9}$, $\cdot\frac{7}{9}$, $.000\frac{4}{5}$, $.00\frac{4}{5}$, $.0\frac{4}{5}$, $\cdot\frac{4}{5}$, etc.

The teacher must give many exercises of this kind and the principle must be brought up for consideration *frequently for weeks to come*. The aim is to teach and enforce that $.0\frac{1}{2}$, $.00\frac{1}{2}$, $.1\frac{1}{2}$, $.2\frac{1}{2}$, $.22\frac{1}{2}$, $.000\frac{1}{2}$, $.012\frac{1}{2}$, $.111\frac{1}{2}$, $.010\frac{1}{2}$, etc. are proper decimal fraction expressions; but $\cdot\frac{1}{2}$, $\cdot\frac{3}{5}$, $\cdot\frac{5}{8}$, $\cdot\frac{9}{12}$, etc. are not decimals, the "." having no effect upon them. The pupils must be made very critical in this matter.

D. Reducing Complex Fractions to Higher Terms

In this step the pupil is given experience in changing complex fractions as follows: $4.3\frac{1}{2}$ to 4.35, $.35\frac{2}{5}$ to .354, $25.13\frac{5}{8}$ to $25.136\frac{1}{4}$ or to $25.1362\frac{1}{2}$ or to 25.13625, $.2\frac{1}{3}$ to $.23\frac{1}{3}$ or to $.233\frac{1}{3}$, etc. In short, it is experience in changing such fractions to the next higher denominator or to the *second* higher equivalent term or to the *third* higher term under the teacher's direction.

1. Written Reductions

$.3\frac{1}{2}$ to hundredths.

$.3\frac{1}{2} = .3$ and $\frac{1}{2}$ of a tenth $= .3 + .0\frac{1}{2}$.

$.1 = \frac{10}{100}$.

$.0\frac{1}{2} = \frac{1}{2}$ of a tenth $= \frac{1}{2}$ of $\frac{10}{100} = \frac{5}{100} = .05$.

$\therefore\ .3\frac{1}{2} = .3 + .05 = .35$.

(The sign $\therefore$ means *therefore*. Pupils should use it regularly).

$.0\frac{1}{3}$ to thousandths.

$.0\frac{1}{3} = \frac{1}{3}$ of a tenth.

$.1 = \frac{100}{1000}$.

$.0\frac{1}{3} = \frac{1}{3}$ of a tenth $= \frac{1}{3}$ of $\frac{100}{1000} = 33\frac{\frac{1}{3}}{1000} = .033\frac{1}{3}$

$\therefore\ .0\frac{1}{3} = .033\frac{1}{3}$.

$.01\frac{5}{6}$ to thousandths.

$.01\frac{5}{6} = .01$ and $\frac{5}{6}$ of a hundredth $= .01 + .00\frac{5}{6}$.

$.01 = \frac{10}{1000}$.

$.00\frac{5}{6} = \frac{5}{6}$ of a hundredth $= \frac{5}{6}$ of $\frac{10}{1000} = \frac{1}{6}$ of $\frac{50}{1000} = 8\frac{\frac{1}{3}}{1000} = .008\frac{1}{3}$.

$\therefore\ .01\frac{5}{6} = .01 + .008\frac{1}{3} = .018\frac{1}{3}$.

Very many exercises in such written analysis work should be given. It gives training in analysis and additional experience in reading and writing complex decimal fractions. The child should have enough of this work to enable him to make the reductions mentally without analysis as in *3* below.

2. *Repetends, Repeating Decimals, Circulating Decimals*

There should be many exercises in this work of reducing complex decimals to higher equivalents and the dictations should include fractional parts in 3ds, 6ths, 7ths, 9ths, 12ths, etc.—$.02\frac{2}{3}$, $.0\frac{1}{6}$, $.00\frac{5}{6}$, $2.01\frac{1}{6}$, $.3\frac{1}{7}$, $.01\frac{5}{7}$, $1.1\frac{1}{9}$, $2.0\frac{1}{15}$, etc. The dictations should sometimes direct that the reductions be carried to millionths or even higher. In this way the pupil will discover that $\frac{1}{3}$ gives a decimal in which 3 is repeated indefinitely, that $\frac{2}{3}$ gives a repeating 6, that $\frac{1}{7}$ gives a repeating 142857, that $\frac{1}{9}$ gives a repeating 1, etc.

The terms "repeating decimals" and "circulating decimals" should be introduced and the pupil should be required to speak of a decimal fraction having a repeating figure or a set of repeating figures as a "repeating" or a "circulating" decimal—circulating because in its endless feature it is like a wheel, a circle.

The term "repetend" may be used in speaking of the part that repeats. The repetend in

$.3\frac{1}{3}$ is 3; in the decimal equivalent of $\frac{2}{3}$, it is 6; in the decimal from $\frac{1}{7}$, it is 142857.

In each reduction that gives a repeating decimal, the child should be required to say, This is a repeating decimal, the repetend is —.

3. Oral Reductions

These are reductions like those in (*1*) above excepting that the work is *mental* and *without analysis.*

$.02\frac{2}{5}$ to thousandths.

The child is required to write the result directly—without written computation.

("$.02\frac{2}{5} = .024$").

$.1\frac{2}{3}$ to 1000ths.

("$.1\frac{2}{3} = .166\frac{2}{3}$").

Many mental exercises of this kind must be given. The pupil should be able to write out such reductions where the denominator of the fraction to be reduced is 12 or less. These exercises should include the reductions of $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, etc.—fractions that give the common repeating decimals. When a reduction gives a repeating decimal, the pupil should so state, as in (2).

E. Reducing Simple and Complex Decimals to Lower Terms

1. Simple Decimals—By Inspection

This step contemplates making reductions mentally, making use of decimals whose reductions are simple. This is very practical mental work and much of it should be given.

$.35$ to 10ths $(.3\frac{1}{2})$.

$.425$ to 100ths $(.42\frac{1}{2})$. To 10ths $(.4\frac{1}{4})$.

$.3475$ to 100ths $(.34\frac{3}{4})$.

3.24 to 10ths $(3.2\frac{2}{5})$.

$.28$ to 10ths $(.2\frac{4}{5})$.

1.389 to 100ths $(1.38\frac{9}{10})$.

Many of these must be given. This is the preparation for (*3*) below.

2. *Complex Decimals—By Inspection*

In this step the dictations are those which result from $\frac{2}{3}$ and $\frac{1}{3}$—$.333\frac{1}{3}$ to 10ths, $333\frac{1}{3}$ to 100ths, $2.6666\frac{2}{3}$ to 100ths, etc. ($.333\frac{1}{3} = .3\frac{1}{3}$, $.333\frac{1}{3} = .33\frac{1}{3}$, $2.6666\frac{2}{3} = 2.66\frac{2}{3}$, etc.) This form of mental work is very practical and much of it should be done.

3. *Written Reductions with Analysis—Complex Decimals*

This section is an application of Chapter XXVI (*J 2*) of Part I.

$.48\frac{1}{3}$ to 10ths.

$48\frac{1}{3} = .4\frac{8\frac{1}{3}}{10}$.

$\frac{8\frac{1}{3}}{10} = \frac{1}{10}$ of $\frac{25}{3} = \frac{25}{30} = \frac{5}{6}$.

$\therefore .48\frac{1}{3} = .4\frac{8\frac{1}{3}}{10} = .4\frac{5}{6}$.

$2.014\frac{5}{9}$ to hundredths.

$2.014\frac{5}{9} = 2.01\frac{4\frac{5}{9}}{10}$.

$\frac{4\frac{5}{9}}{10} = \frac{1}{10}$ of $\frac{41}{9} = \frac{41}{90}$.

$\therefore 2.014\frac{5}{9} = 2.01\frac{4\frac{5}{9}}{10} = 2.01\frac{41}{90}$.

$1.010\frac{3}{5}$ to 100ths.
$1.010\frac{3}{5} = 1.01\frac{\frac{3}{5}}{10}$.
$\frac{\frac{3}{5}}{10} = \frac{1}{10}$ of $\frac{3}{5} = \frac{3}{50}$.
$\therefore\ 1.010\frac{3}{5} = 1.01\frac{\frac{3}{5}}{10} = 1.01\frac{3}{50}$.

Many exercises like these should be given.

SUMMARY.

1. The first aim of this chapter is to teach the child that a fraction space or position can not be "occupied" by a common fraction; that "empty" or "occupied" is indicated by one of the ten numeral signs, never by a common fraction; and that, consequently, a decimal point before a common fraction (see p. 596) ($.\frac{1}{2}$, $.\frac{3}{4}$, etc.) does not signify 10ths. (*A*, *B*, and *C*.)
2. Exercises in reducing complex decimals to higher forms simple or complex. This is written work and the reduction process is analytical. (*D*.)
3. Mental reductions of complex decimals to higher forms without analysis. (*D*, *3*.)
4. Mental exercises in reducing complex decimals to lower forms—without analysis. (*E*.)
5. Exercises in reducing complex decimals to lower forms. This is written and analytical. (*E*, *3*.)

Time required for this work, including necessary reviews, ten days.

CHAPTER XI

DIVISION OF DECIMALS

Topic:—The Development of "Pointing off"

A. Aim

In this subject there is but one problem—that of teaching how to "point off" the quotient. The process of dividing is like that with whole numbers. Aside from "pointing it off" the pupils will find the quotient without instruction of any kind.

The aim of this development work, however, is not expressed in terms of "pointing off" but in terms of "stopping place" of integers. The child is to learn to determine where the integral part, if any, of the quotient ends and the fractional part, if any, begins. The development purpose is not to teach to "point off" the quotient but to find in it the dividing line between the integral and the fractional parts. This being determined the pupil places there the "stopping place" sign—the decimal point.

The teacher must have in view, therefore, in this work the following aims:

(1) To develop the power to determine in

each exercise in what stage of the division process the quotient figures begin to be fractional, and

(2) To develop in the pupil the habit of determining this and indicating it by written sign before the operation of dividing is begun.

B. The Teaching Steps

1. Finding the Quotient Decimal Point After Dividing

These are exercises in finding and marking the place where, in the division process, the integral part of the quotient ends and the fractional part begins.

(*a*) $.63 \div .3 =$. The teacher writes this on the blackboard.

NOTE 1.—The following questions should be asked after writing the exercise as above and before the work upon it begins—

"What kind of a number is the dividend?" ("Fractional.")

"What kind the divisor?" ("Fractional.")

"Read the dividend the long way." ("6 tenths and 3 hundredths.")

"Read *it the short way*." ("Sixty-three hundredths.")

"Each of you may perform the division."

Each child writes the exercise $.63 \div .3 =$ but must be required to perform the division operation in regular way—$\begin{array}{r} 21 \\ \hline .3).63 \end{array}$. Every child will write the quotient 21 but there will be all kinds of uses with it of the decimal point.

A child should now be sent to the blackboard to write what he has on his paper—$.63 \div .3 =$ $\begin{array}{r}\underline{21}\\ .3).63\end{array}$ and then should be required to *prove* the position of the decimal point in the quotient. The *proof* is by the *common fraction* process. *He is told* that he can find out what the quotient is by solving as a common fraction as $\frac{63}{100} + \frac{3}{10} = \frac{63}{100} + \frac{30}{100} = 2\frac{3}{30} = 2\frac{1}{10} = 2.1$.

This analysis and proof the pupil should be able from his common fraction experience to make unaided.

From this analysis he proves that the 2 of the quotient is integral and that the 1 is fractional—.1.

The work by the child at the blackboard and by each child on his paper at his desk should be in form as follows:

(*x*) $.63 \div .3 =$

(*y*) $\begin{array}{r}\underline{2.1}\\ .3).63\end{array}$ (This quotient is pointed off *after* the proof in (2).)

(*z*) $\frac{63}{100} + \frac{3}{10} = \frac{63}{100} + \frac{30}{100} = 2\frac{3}{30} = 2\frac{1}{10} = 2.1$.

This is the form that *every* exercise by the pupils under this subject should take—

(1) The exercise *stated* as in (*x*);
(2) The *operation* form as in (*y*);
(3) The analysis *proof* as in (*z*).

The following questions should be asked *after* each exercise:

"What kind of a number is the quotient?" ("Mixed.")

"In what part of the division did we get the integral part of the quotient?" ("When we divided the .6.")

"In what part of the division did we get the fractional part?" ("When we divided the .03.")

We will now, *after the analysis* (z), mark the dividend in the real division work to separate that which gave us the integral part of the quotient from that which gave us the fractional part.

$$\begin{array}{r} 2.1 \\ .3\overline{).6_{\wedge}3} \end{array}$$

NOTE 2.—In all (y) operations the pupil must write each quotient figure directly above the dividend figure that gave it so that the decimal point comes *directly above the caret.*

The teacher completes the questioning by showing that in this exercise the divisor is 10ths and the part of the dividend that gave the integral part of the quotient is 10ths—10ths divided by 10ths gave an integral quotient.

(*b*) $2.46 \div .2 =$

The work with this follows the same line as that with (*a*)—the questions before and after and the three solution steps (*x*), (*y*), and (*z*). In this exercise, besides asking the pupil the kind of number used as dividend, the kind as divisor, and to read the dividend long way and short way, the teacher should ask him to read the dividend as *a fraction*—two hundred forty six hundredths.

When the work is finished—division, proof, and question—the written work will be marked as before to set off the part of the dividend that gave integral quotient figures from that which gave fractions—

$$\begin{array}{r} 12.3 \\ .2\overline{)2.4\underset{\wedge}{}6} \end{array}$$

The teacher will also bring out, as before, that 10ths divided by 10ths gives an integral quotient. Another question to be asked regularly is, "What does this mark ($\wedge$) show?" The pupil's answer is, "It shows how far in the division I get an integral quotient."

(*c*) 2.046 ÷ .02 =

This is to be used as outlined for (*a*) and (*b*).

$$\begin{array}{r} 102.3 \\ .02\overline{)2.04\underset{\wedge}{}6} \end{array}$$

At the conclusion of the solution it will be brought out that in this exercise we have 100ths divided by 100ths up to the caret ($\wedge$) and that 100ths divided by 100ths gives an integral quotient.

(*d*) Other exercises of the same kind. The quotients in these exercises are mixed numbers or fractions having denominations in 10ths or 100ths.

5.5 ÷ 5 =
24.2 ÷ 2 =
.0369 ÷ .003 =
.224 ÷ .07 =
.345 ÷ .015 =*
.264 ÷ 1.1 =
3.84 ÷ 1.2 =

21.7 ÷ .7 =
18.06 ÷ .6 =
18.909 ÷ .9 =
3.368 ÷ .8 =
30.8 ÷ 1.4 =*
.0495 ÷ .15 =
.672 ÷ 2.1 =*

.55 ÷ .5 =
36.903 ÷ .3 =
.0048 ÷ .002 =
2.52 ÷ 1.2 =
.0416 ÷ .013* =
.0512 ÷ .16* =
7.01 ÷ 2.2* =

NOTE 3.—During these exercises the teacher should formulate into one statement the results of the child's experiences with integral quotients as follows: I have an integral quotient when I divide wholes by wholes, or 10ths by 10ths, or 100ths by 100ths, or 1000ths by 1000ths, etc. I set off in the dividend the part that has units like those in the divisor.

This statement should be given again and again by the pupils, and every day. It is very important because it is the principle involved in placing the caret. When in the statement as above he says, "I set off, etc.," the pupil inserts the caret (∧) in its proper place or points to it if it has already been placed.

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* The (*y*) form in such exercises will be in "long division" form. The teacher must see that the suggestion in Note 2 above is *strictly* observed. At the conclusion of each solution in this and in (*e*) below the teacher must bring out what the character of the division up to the caret is—whether wholes by wholes, 10ths by 10ths or 100ths by 100ths, etc., as in (*c*) above.

(*e*) Exercises in which the quotients are fractions whose terms are higher than 10ths. This is a step by itself because it is not always easy for a pupil to think of *fractions beginning* at a place in the division operation where there is no quotient figure. In .0036 ÷ .3 =, the caret (∧) in the division work is as follows: $3\overline{)0\underset{\wedge}{0}.36}$. There is no quotient figure till the operation reaches the .003. The pupil needs experience in finding the caret's position in such exercises.

.0299 ÷ .13 =
.0576 ÷ 1.8 =
.264 ÷ 1.1 =
.0768 ÷ 2.4 =
.00015 ÷ .03 =
.0012 ÷ .06 =
5.621 ÷ .5 =
.00688 ÷ .16 =
.00598 ÷ 1.9 =

.000396 ÷ .009 =
.0014 ÷ .7 =
.00009 ÷ .009 =
.028 ÷ 1.4 =
.000112 ÷ .016 =
.000328 ÷ .08 =
.0121 ÷ 1.1 =
.00387 ÷ .09 =
.00002 ÷ .002 =

The suggestions in Note 2 above must not be overlooked.

2. *Finding the Quotient Decimal Point Position in Advance*

These are exercises in finding *in advance* of the operation of division where, in the division process, the integral part of the quotient will end and the fractional part begin. In other words, the child when he writes an exercise in the (y) form must at once, before performing the division operation, insert the caret.

In these exercises the three forms of expression—(x), (y), and (z)—as suggested in (*1 a*) above, must be shown, the form in (z) being the *proof* that the quotient is right numerically and that the decimal point is correctly placed.

The child's order of solutions must be—1st, the (x) form of statement of the exercise; 2d, the (y) form of statement; 3d, the insertion of the caret (∧) at its proper place in the dividend *before dividing* (thus anticipating the caret's position); 4th, the complete operation in (y) with the decimal point in its proper place as indicated by the caret; and 5th, the (z) analysis as proof that the caret position was or was not correctly anticipated and through it the decimal point properly placed. The questions in *1* and the suggestion in Note 3 above must not be overlooked.

The child *must* also state, when he places each caret, the character of the division to that point—whether it will be wholes by wholes, or tenths by tenths, or 100ths by 100ths, etc. This statement, *required unvaryingly*, will develop a habit of looking for the character of the division up to the caret. There will *always* be a division of wholes by wholes, 10ths by 10ths, 100ths by 100ths, or 1000ths by 1000ths, etc., but in the division there may not be an integral quotient. .0015 ÷ .5 = will have the (y) form as follows: $5\overline{)\,.0_{\wedge}015}$. In this the division to the caret will be 10ths by 10ths. The quotient will be 0.003—no integers, no 10ths, no 100ths.

(*a*) Exercises in Anticipating the Caret's Position.

.368 ÷ .08 =
.0455 ÷ .07 =
5.35 ÷ 5. =
21.069 ÷ .03 =
.03168 ÷ .009 =
18.036 ÷ 18. =
11.055 ÷ .5 =
6.012 ÷ .006 =
.0021 ÷ .07 =
.00126 ÷ .03 =
.0026 ÷ 1.3 =
.01575 ÷ 1.5 =
.01664 ÷ .16 =
.3528 ÷ .007 =
12.006 ÷ 6. =
15.28 ÷ .08 =
14.0021 ÷ .0007 =
.0792 ÷ .0024 =

.0038 ÷ 1.9 =
.00024 ÷ .012 =
1.2012 ÷ 1.2 =
.13026 ÷ .13 =
.02604 ÷ 1.2 =
214.05 ÷ 5. =
45.018 ÷ .009 =
12.05 ÷ .09 =
6.88 ÷ 16 =
.06015 ÷ .15 =
.00816 ÷ .8 =
.01176 ÷ 1.2 =
.0017 ÷ .013 =
.00135 ÷ .15 =
17.12 ÷ 8. =
33.044 ÷ .011 =
16.008 ÷ .008 =
.144 ÷ .016 =

Observe carefully the suggestion in Note 2 above, and that in Note 4.

In each exercise above, the statement by the child as he inserts the caret, "This will be wholes divided by wholes"; or, "This will be 10ths divided by 10ths;" etc.; is very important. In .368 ÷ .08 = he will (in the (*y*) form) place the caret—$.36_{\wedge}8$—and will say, "This will be 100ths divided by 100ths. In 18.036 ÷ 18. =, the caret will be placed—$18\underset{\wedge}{.}036$—and he will speak of the division as wholes by wholes. In 6.012 ÷ .006 =, he will place the caret $6.012_{\wedge}$ and will speak of the division as 1000ths by 1000ths. In .0021 ÷ .07 =, he will place the caret—$.00_{\wedge}21$—. In the oral work of his division operation he will speak of the quotient as having no integers, no 10ths, 3 hundredths—0.03.

(*b*) Exercises in Anticipating the Position of the Caret—Dividends and Quotients Fractional.

The aim of these exercises is to give the pupil experience with solutions where the caret is at the decimal point and the dividend a fractional number. It offers no great difficulty; yet it is, to the child, a somewhat strange situation and requires direct and repeated experiences. In this work the caret will mean, This is the place where the fraction spaces (abacus) of the quotient begin. The child's *first act* in the division operation after placing the caret should be to place above it the decimal point. He should then state the character of the division work up to the caret—whether it is wholes by wholes, 10ths by 10ths, or 100ths by 100ths, etc.

Suggestion.—The teacher *must insist* from now on that the *first* act in *every* division operation ((*y*) form) *must be* to place the *caret*, the *second* act to place the *decimal* point above the caret, the third act to state the character of the division up to the caret. This must become a habit if we would secure accuracy.

The child should learn to look for a division of wholes by wholes, or 10ths by 10ths, or 100ths by 100ths, etc., in every exercise. He must learn that every exercise will require the caret and that consequently there will be a division of *like units*—wholes by wholes or 10ths by 10ths or 100ths by 100ths, etc. He will learn that such division may give an "*empty*" quotient (0), or it may give an integral quotient.

.0369 ÷ 9. =
.2892 ÷ 12. =
.0006 ÷ 3. =
.0264 ÷ 6. =
.0406 ÷ 7. =
.12036 ÷ 12. =
.009018 ÷ 9. =
.0132 ÷ 11. =
.1401 ÷ 3. =
.1111 ÷ 11. =

.00461 ÷ 9. =
.1012 ÷ 11. =
.1188 ÷ 12. =
.1278 ÷ 9. =
.1356 ÷ 12. =
.00012 ÷ 6. =
.01016 ÷ 8. =
.10101 ÷ 7. =
.00208 ÷ 8. =
.00108 ÷ 9. =

The habit developed is better if the child in the solution of exercises like the above—dividends that are not mixed numbers—places a cipher (empty) in the first integral space when he inserts the caret. For illustration, when dividing .0369 by 9, he writes the (*y*) form as follows:—

$9.\overline{)0\underset{\wedge}{\cdot}\overset{\cdot}{0}369}$; in dividing .0006 by 3, he writes,

$3\overline{)\overset{\cdot}{0}\underset{\wedge}{\cdot}0006}$

3. *Reduction of Dividend to Higher Equivalent Necessary Sometimes*

In dividing 36 by .4 there is no opportunity in the present form of the dividend to divide wholes by wholes, 10ths by 10ths, or 100ths by 100ths, etc. The division on that account is impossible. The exercise requires us to divide by 10ths (.4). In order to do this there must be 10ths to divide. Hence the reduction of the dividend to 10ths.

$36 = 36.0$	This shows the change to 10ths.
$36.0 \div .4 =$	This is the (*x*) form.
$.4\overline{)36\cdot\underset{\wedge}{0}}$	This is the (*y*) form.

By this change the 36 becomes 360 tenths. It is possible to divide 360 tenths by 4 tenths. The division is now 10ths by 10ths.

In the same way, $4.8 \div .02$ is not possible because the requirement is to divide by 100ths (.02) and the dividend does not have 100ths.

By reducing this dividend to 100ths the operation becomes possible—

(1) 4.8 = 4.80. (This shows the change to 100ths.)

(2) $4.80 \div .02 =$ ((*x*) form.)

(3) $\begin{array}{r} 240. \\ .02\overline{)4.80}_{\wedge} \end{array}$ = ((*y*) form.)

(4) $\frac{480}{100} \div \frac{2}{100} = 240.$ ((*z*) form.)

The form of solution shown above must be followed: 1st, a statement of the reduction; 2d, the (*x*) form; 3d, the (*y*) form; 4th, the (*z*) form, if the *proof* is desired. The (*z*) form may be omitted unless specifically called for, if desired.

(*a*) Exercises Giving Integral Quotients.

$35. \div .7 =$
$28. \div .07 =$
$4.27 \div .007 =$
$3.6 \div .06 =$
$48.6 \div .06 =$
$27. \div .09 =$
$13.39 \div .013 =$
$21.035 \div .0007 =$
$.24 \div .008 =$
$44. \div .011 =$
$9.36 \div .009 =$
$328. \div .08 =$
$360. \div .18 =$
$108. \div 1.2 =$

$14.4 \div .012 =$
$38.19 \div .019 =$
$.00015 \div .0000015 =$
$.02 \div .0002 =$
$21.14 \div .0014 =$
$.299 \div .0013 =$
$480.16 \div .00016 =$
$281.4 \div .014 =$
$.255 \div .0005 =$
$.196 \div .0008 =$
$151.2 \div .0012 =$
$.132 \div .0012 =$
$.171 \div .0009 =$
$.01012 \div .000011 =$

(*b*) Exercises Giving Fractional Quotients.

In these exercises, the reduction of the dividend is due to the division not being "*even.*" In exercises like those in (*a*) the reduction was necessary *before* the division operation. In these exercises the reduction will be found necessary *after* the (*y*) form operation begins. In (*a*) the reductions were to make division possible. In these exercises the reduction is to avoid a *complex* decimal fraction quotient.

The aim of these exercises is to give experience in the reduction of the dividend after it is found that *even* division is impossible without it.

The following illustrates the form of the work:—$1.23 \div .6 = \begin{array}{r} 2.0 \\ .6\overline{)1.2\underset{\wedge}{\,}3} \end{array}$

There will be at this point a remainder. The quotient, if the dividend is not changed, will have a complex fraction ($2.0\frac{1}{2}$). The child must avoid this by a further reduction of the dividend. This reduction should be made in the first few solutions by beginning the (*y*) form over again. His written work will now appear as follows:

$1.23 \div .6 =$ ((*x*) form).

(i) $\begin{array}{r} 2.0 \\ .6\overline{)1.2\underset{\wedge}{\,}3} \end{array}$ ((*y*) form).

(ii) $\begin{array}{r} 2.05 \\ .6\overline{)1.2\underset{\wedge}{\,}30} \end{array}$ ((*y*) form).

The object in having the pupil rewrite the solution, as in the (y) forms above, (i) and (ii) is to give him an opportunity to learn through his own experiences that the reduction of the dividend to higher terms affects only the quotient figures from the part of the division operation which used the annexed ciphers.

Later instead of repeating the whole operation with each reduction, he will do as we do—annex ciphers to his dividend as the operation proceeds and he finds the need of one and perhaps another and another reduction for the sake of an "even" division.

Note 4.—The teacher must be very careful about her own language in this matter of annexing ciphers. She must use the term, "Change to next higher term of the series of equivalents," or "Change to the 2d higher term, etc.," or "Change to the 3d higher term, etc." The child must use the language of changing to "higher terms of the series of equivalents." Otherwise, he loses sight of the fact that the annexed cipher effects no change in the quantity expressed by the dividend.

The *oral work* in connection with changing an expression like $.6\overline{)1.2\underset{\wedge}{\dot{3}}}$ above to $.6\overline{)1.3\underset{\wedge}{\dot{2}}0}$ should be, "I must change the 123 hundredths to the next higher equivalent, 1230 thousandths." This form of oral statement must be insisted upon.

(1) Exercises in Which the Pupil Repeats the (*y*) Statement.

In the following exercises the solutions should be written as in the solution shown above—with the (*y*) form repeated when the reduction of the dividend is made:

.36 ÷ .8 =
12.8 ÷ .05 =
150.33 ÷ 15. =
44.8 ÷ .7 =
4.73 ÷ .22 =
3.69 ÷ .18 =
.427 ÷ .014 =
1.59 ÷ .012 =*
1.48 ÷ .05 =
.847 ÷ .028 =

27.3 ÷ .6 =
1.8 ÷ .08* =
31. ÷ .4* =
3.3 ÷ 2.2 =
.48 ÷ 3.2 =
20 ÷ 1.6* =
3.1 ÷ .25* =
.35 ÷ 1.4 =
.726 ÷ .24 =
.063 ÷ .06 =

(2) Exercises in Which the (*y*) Form is not Repeated.

The pupil is now taught to make reductions in the dividend when necessary without delaying or changing his division work. In annexing the cipher, however, the oral statement suggested in Note 4 above must not be overlooked. In dividing, for illustration, 15.6 by .5, the (*y*) form is $.5\overline{)5.\dot{6}\underset{\wedge}{.}}$ When his division reaches the form $\begin{array}{r}11.\\ .5\overline{)5.6_{\wedge}}\end{array}$ he says: "The division is not 'even,' therefore I reduce the 56 tenths to the next higher equivalent, to hundredths." He now annexes the cipher and says, "I now have 560 hundredths." The division proceeds now in the regular way— $\begin{array}{l}11.2\\ 5\overline{)5.6_{\wedge}0}\end{array}$. If it should be necessary to make a second reduction (annex a second cipher), the same form of work is continued. For illustration, in $.19 \div .4 =$, the child makes the solution $.19 \div .4 = ((x)$ form).

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* In these solutions the pupil must make one reduction to make a division possible as in (*a*) above, and a second reduction to make the division *even*.

$$\begin{array}{r}.4\\ .4\overline{).1_{\wedge}9}\end{array} \quad ((y) \text{ form}).$$

The division is not even, therefore I reduce the 19 hundredths to the next higher equivalent—to thousandths." The cipher is annexed $\begin{array}{r}.4\\ .4\overline{).1_{\wedge}90}\end{array}$ "I now have 190 thousandths." The division proceeds— $\begin{array}{r}.42\\ .4\overline{).1_{\wedge}90}\end{array}$ "The division is not even, therefore I reduce the 190 thousandths to the next higher equivalent—to 10-thousandths." "I now have 1900 10-thousandths."

$$\begin{array}{r}.42\\ .4\overline{).1_{\wedge}900}\end{array} \qquad \begin{array}{r}.425\\ .4\overline{).1_{\wedge}900}\end{array}$$

This completes the operation.

The following are exercises for practice:

$.141 \div .6 =$
$2.8 \div .05 =$
$7.707 \div .35 =$
$.99 \div .45 =$
$1.14 \div 1.2 =$
$2.9 \div .08 =$
$3.5 \div .04 =$
$.27 \div .6 =$

$37. \div 1.6 =$
$1.8 \div .24 =$
$2.1 \div .28 =$
$4. \div 3.2 =$
$27.5 \div 2.2 =$
$1.35 \div 1.8 =$
$.105 \div 1.4 =$
$1.4 \div .16 =$

4. Repeating Decimals—Quotients Exact and Quotients Approximate

a. QUOTIENTS EXACT

These exercises require no development or instruction because, aside from the use of the decimal point, they do not differ from exercises in division of integral numbers. The aim is to accustom the child to finding mixed number quotients with fractional part limited to some stated denominator—to 10ths, or to 100ths, or to 1000ths, etc.

The dictations are to be given in all cases with this fraction limit stated.

To thousandths.

$.21 \div .9 =$
$1.33 \div .06 =$
$71.8 \div .15 =$
$.0437 \div .013 =$

To hundredths.

$39.2 \div .13 =$
$123. \div .11 =$
$.589 \div 1.4 =$
$.0146 \div .12 =$

To thousandths.	To ten-thousandths.
32.1 ÷ .09 =	.0126 ÷ 1.7 =
.0001 ÷ .00002 =	345 ÷ .007 =
.06 ÷ .007 =	190.1 ÷ .03 =
1. ÷ .002 =	256. ÷ .12 =

b. QUOTIENTS APPROXIMATE

In .4 ÷ .6 =, the exact quotient to 1000ths is $.666\frac{2}{3}$; the approximate quotient to the same term is .667 −. In .05 ÷ .7 =, the exact quotient to 1000ths is $.071\frac{3}{7}$; the approximate quotient is .071 +.

The aim of this step is to introduce the term *approximate*, and then its opposite *exact*. The teacher must put these two terms into the regular working vocabulary of the class. Exercises must be dictated as "approximate to 100ths" or "exact to 100ths" until pupils use them freely. The teacher, instead of avoiding the use of approximate, must seek to use it whenever its use is appropriate.

With the term approximate must go the signs + and −. The pupil must learn to use these signs with approximations. He must learn to write .67 − instead of .66 + as the approximate quotient, to 100ths, of .2 ÷ .3 =. He must learn that the proper approximate quotient of 1 ÷ 3 =, to 100ths, is .33 + and that the quotient of .5 ÷ .9 =, to 100ths, is either

$.56-$ or $.55+$ as one prefers. In brief, the pupil must learn to make use of, and proper use of, $+$ and $-$ in these approximations.

Exercises in Approximate Quotients.

To thousandths.	To thousandths.
$.05 \div .7 =$	$218. \div .09 =$
$1.12 \div .3 =$	$15.6 \div .007 =$
$816.1 \div .06 =$	$3.89 \div 1.12 =$
$.0808 \div .07 =$	$10.101 \div .011 =$
To 10-thousandths.	To hundredths.
$.0305 \div .00009 =$	$.0001 \div .003 =$
$2.101 \div .06 =$	$.0101 \div .03 =$
$71.02 \div .003 =$	$.0502 \div .007 =$
$.0202 \div .03 =$	$.001 \div .011 =$

The best results in the use of the two words is secured by requiring the pupil, when solving for approximate quotients, to write with each of such quotients the *exact* quotient. For illustration, in finding the approximate quotient of $.06 \div .07 =$, to 100ths, the child writes—

$$\begin{array}{l} .06+.07 = \\ \underline{.857}+ \quad (.857\tfrac{1}{7} \text{ exact.}) \\ .07\overline{).0_{\wedge}6000} \quad (.857+ \text{ approx.}) \end{array}$$

SUMMARY.

1. The aim of this chapter—

(1) To train the pupil to determine *after* the operation and to mark the

point in the division process where the integral part of the quotient ends and the fractional part begins; and,

(2) To discover and state the principle involved (the character of the divisions that give integral quotients); and,

(3) To train pupils to apply this principle in determining *in advance* and in marking with a proper sign this "stopping place" point.

2. Exercises in dividing in which the position of the decimal point in the quotient is proved by common fraction solution. Each division process is examined—

   (1) To find and mark the part of the dividend that yielded the integral part of the quotient.

   (2) To enable the pupil to discover that 10ths divided by 10ths, 100ths by 100ths, 1000ths by 1000ths, etc. give integral quotients, just as in common fraction division (long form).

3. Exercises in marking the "stopping place" point in advance, this being the

point in the dividend up to which the division will be 10ths by 10ths, 100ths by 100ths, or 1000ths by 1000ths, etc., using—

(1) Exercises in which it is not necessary to change the denominator of the dividend;

(2) Exercises in which it is necessary to reduce the dividend to higher terms —annex ciphers.

(3) Exercises in which the quotients are repeating decimals:
  (*a*) Quotients exact.
  (*b*) Quotients approximate.

Time required for this work, including necessary reviews, ten days.

CHAPTER XII

SHORT METHODS AND ANALYSES

Topics:—Operations with 10's and 100's. Analysis of "of Work." Equations

A. Aim

Now that the "pointing off" in multiplication and division has been developed, exercises should be given in short methods in these operations with 10's and 100's—by changing the position of the decimal point.

This development should be conducted in class exercises in which one child and then another is sent to the blackboard for a solution with full oral statement of every step. This oral work must be analytical in order that the operations may have a basis in reason.

The exercises are, of course, very simple. The pupil would have no difficulty in solving them by the ordinary processes. The teacher's problem is (1) to teach the pupil to make a *very brief* horizontal solution and (2) to give the proper oral statements.

B. Multiplication

1. The Form of the Analytical Solution

$10 \times .004 =$

The teacher writes it on the blackboard and calls a pupil for the solution. The child's written work must be in form as follows:

$10 \times .004 = .040 = .04$

His oral work given as he writes on the blackboard, each part spoken in connection with the corresponding written work, should be—

("Ten times four thousandths is forty thousandths"—$10 \times .004 = .040$.

"Forty thousandths is the second term of the series of equivalents from four hundredths. I change the forty thousandths to its lowest terms and have four hundredths. My product is four hundredths.")

It will be necessary for the teacher to guide the pupil to this form of horizontal statement and to the oral language as suggested above.

$10 \times .07 =$. (On the blackboard.)

The horizontal form of written statement is—$10 \times .07 = .70 = .7$.

The oral language should be, "Ten times seven hundredths is seventy hundredths. The seventy hundredths is the second term of the series of equivalents. I change it to its lowest terms, seven tenths. Seven tenths is my product."

$100 \times .0003 = .0300 = .03$.

The oral language with this written work is to the effect that the .0300 is the third term of the series of equivalents, and that the lowest terms and the product is .03.

The following exercises should be given:

100 × .005 =
100 × .04 =
10 × 2.03[*] =
10 × .007 =
100 × .0004 =
10 × 1.8 =
1000 × .0006 =
100 × 1.005 =
10 × 2.12 =

2. *Shortening the Written Analysis*

In this step the analysis work is shortened to the simple statement with the product. The *oral* work *continues unchanged.*

10 × .09 = .9.

In solving this the pupil's oral statement is as follows: "Ten times 9 hundredths is 90 hundredths. 90 hundredths is the second term of the series of equivalents. My product—the lowest terms—is nine tenths." The reduction to lowest terms is, of course, *mental.*

Exercises for this short analysis and the oral work:

10 × 1.2 =
100 × 3.4 =
10 × 4.1 =
10 × .051 =
100 × .006 =
10 × 2.1 =
100 × .15 =
10 × 1.4 =
10 × 1.01 =
100 × 1.11 =
10 × 2.5 =
100 × 2.5 =
1000 × .2 =
100 × .5 =
100 × .8 =
10 × .4 =

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[*] The oral work is, Ten times two hundred three hundredths is two thousand thirty hundredths (20.30), etc.

10 × .61 =
10 × .04 =
100 × .18 =
1000 × .09 =

100 × .05 =
100 × .03 =
100 × 1.3 =
106 × 1.6 =

These with other similar exercises should be given until the pupil is able to write the statements and give the oral work with a fair degree of rapidity. The pupil should solve many problems of this kind mentally.

3. Analytical Written Solutions

This step relates to all forms of multiplication in which the work is written—solutions in which the multiplier may be other than 10, 100, etc.

.8 × .35 =
20 × 1.6 =
.015 × 40 =

2.82 × 50 =
1.5 × .18 =
3.2 × .25 =

.42 × .15 =
5.6 × 3.5 =
.85 × 1.8 =

Each of the products in these numbers ends with a cipher and is therefore the second or third term of the series of equivalents and should be reduced to its lowest terms.

The following illustrates the form of the written work:

$$\begin{array}{r} 3.5 \\ .8 \\ \hline 2.80 = 2.8 \end{array} \qquad \begin{array}{r} .18 \\ 1.5 \\ \hline 90 \\ 18 \\ \hline .270 = .27 \end{array}$$

The pupils should learn to recognize such products as higher terms of series of equivalents and should reduce them as indicated.

C. Division

1. The Long Forms of Solutions

Analysis in multiplication guides the pupil in his thinking. In division, where the dividend is quantitatively less than the divisor, it makes for confusion except in solutions in

$$\text{“of”—}\tfrac{1}{10}\text{ of } .04 = 10\overset{.004}{\overline{)\underset{\wedge}{.}040}} = .004$$

Analysis, therefore, is not attempted. It is not a means to an end.

The form of the solution in “of” exercises is as shown above. In regular division (÷) it is the same except in the first statement—

$$.04 \div 10 = 10\overset{.004}{\overline{)\underset{\wedge}{.}040}} = .004.$$

The teacher will observe that these are the only forms of statement which give rise to division operations.

$$\tfrac{1}{10}\text{ of } .004 = 10\overset{.0004}{\overline{)\underset{\wedge}{.}0040}} = .0004$$

The pupil must be taught this form of solution. He will not fail to observe the form of the quotient .0004 as compared with that of the dividend .004.

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad .0106^{*} \\ \frac{1}{100} \text{ of } 1.06 = 100\overline{)1.0600} = .0106 \\ \qquad\qquad\qquad\qquad\quad \wedge \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad .00105^{*} \\ .105 \div 100 = 100\overline{).10500} = .00105 \\ \qquad\qquad\qquad\qquad \wedge \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad .00015 \\ .15 \div 1000 = 1000\overline{).15000} = .00015 \\ \qquad\qquad\qquad\qquad \wedge \end{array}$$

Additional exercises as follows:—

$2.1 \div 10 =$
$.34 \div 10 =$
$\frac{1}{10}$ of $1.6 =$
$\frac{1}{10}$ of $1.2 =$
$.24 \div 100 =$
$.36 \div 10 =$

$\frac{1}{100}$ of $.012 =$
$\frac{1}{10}$ of $.09 =$
$.109 \div 10 =$
$1.8 \div 100 =$
$\frac{1}{10}$ of $.28 =$
$\frac{1}{100}$ of $.27 =$

$\frac{1}{10}$ of $2.2 =$
$\frac{1}{1000}$ of $.25 =$
$\frac{1}{1000}$ of $.2 =$
$.3 \div 10 =$
$\frac{1}{1000}$ of $.31 =$
$.32 \div 1000 =$

2. *The Short Forms of Solutions*

This step differs from the "long form" work above in that the pupil leaves out the second step.

$.04 \div 10 = .004$
$\frac{1}{10}$ of $.3 = .03$
$.12 \div 100 =$

$\frac{1}{100}$ of $2.3 =$
$\frac{1}{1000}$ of $1\frac{2}{3} =$
$2.4 \div 100 =$

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* If the pupil wishes to find the quotient by long division, there is no reason why he should not do so. In that case this statement would be the full long division solution instead of the short division as here.

Many exercises should be given until the pupil is able to write the statements rapidly. The exercises to be used are like those in *1* above. Much mental work of this kind should be given.

D. Mixed Exercises—Multiplication and Division

These should be given in the short form, the multiplication exercises to be with oral analysis as in *B* 2 above. The work repeats *B* 2 and *C* 2 above.

$.005 \div 10 =$
$10 \times .005 =$
$100 \times .012 =$
$.012 \div 100 =$
$\frac{1}{100}$ of $1.03 =$
$\frac{1}{10}$ of $1.1 =$
$.004 \div 10 =$
$10 \times 1.2 =$
$1.5 \div 10 =$
$.18 \div 10 =$
$10 \times .18 =$
$10 \times 2.1 =$

$1.1 \div 10 =$
$.12 \div 100 =$
$\frac{1}{100}$ of $1.5 =$
$100 \times 1.5 =$
$1000 \times .011 =$
$.011 \div 10 =$
$\frac{1}{10}$ of $.05 =$
$10 \times .05 =$
$100 \times .05 =$
$1.6 \div 10 =$
$1.6 \div 100 =$
$\frac{1}{10}$ of $1.6 =$

Changing Multiplicand into Product and Dividend into Quotient

This is the view that the pupil must acquire. We ordinarily speak of this operation as moving the decimal point. We do not move the decimal point. We, to save time and to save writing, change a number which expresses the multiplicand so that it expresses the product instead; or, we change the number expressing the dividend so that it expresses the quotient instead. In either case, we use the figures of the old expression to make a new expression—a quotient or a product. The child must be given this view of it. In other words, he must learn to think of the operation as one in which figures expressing a multiplicand or a dividend are changed without rewriting so that they express a product or a quotient.

1. *Exercises in Multiplication—Oral Dictation*

These dictations are given in the form, Write 21 hundredths. (.21). Now multiply it by 10 by changing it to the product. (2.1). Write 3 and 5 tenths. (3.5.) Multiply by 100 by changing it to the product. (350.) In each case the questions must be asked, Which way did you move the point in changing to the product? How far? The question, How far? is answered in terms of spaces—one space, two spaces, etc.

Write 6.2. Multiply by 100.
Write .034. Multiply by 10.
Write 2.95. Multiply by 1000.

Many dictations of this kind should be

given. Exercises of this kind with the question, How far? should be given occasionally for months.

2. Exercises in Division—Oral Dictation

Exercises of the same kind and following the same plan should be given in division. The questions, Which way did you move the decimal point in making the quotient? and, How far? must be used regularly. Exercises like these must be given for months. Many problems of this kind should be given for entirely mental work.

3. Mixed Exercises—Oral Dictations

These are multiplication and division mixed. The question, Which way, etc.? and How far? must be used regularly.

4. Rules or Statements

Statements must be formulated and pupils must recite them. ("When we make a product we move the decimal point to the right." "When we make a quotient we move the decimal point to the left." "Moving the decimal point one space to the right makes a product by 10." "Moving the decimal point two spaces to the right makes a product by 100." "Moving the decimal point three spaces to the right, etc." "Moving the decimal point one space to the left makes a quotient by 10."

"Moving the decimal point two spaces," etc.)

The teacher should formulate statements (brief) covering all these points and pupils must learn them and recite upon them frequently.

F. Partition with Decimals—"Of Work"

This should be entirely analytical for many weeks, the development being confined to analytical work and the exercises that are given from time to time later being given in the same form. The child, in fact, will use partition with decimals either in exercises or in applied problems in the analytical form through the elementary school. It is an operation which demands written or mental analysis if the student is to rely upon himself for accurate results. The questions suggested in Part II, Chapter XXVI, *F 1* and *2* for similar work with common fractions should be used with each exercise. After a few exercises the pupil should be able to ask and answer the questions without suggestion.

Form of written work.

(i) $.02$ of $12 =$
$.01$ of $12 = 12 \div 100 = .12.$
$.02$ of $12 = 2 \times .12 = .24$

(ii) $.6$ of $1.3 =$
$.1$ of $1.3 = 1.3 \div 10 = .13$
$.6$ of $1.3 = 6 \times .13 = .78$

(iii) .00012 of 25.6 =

.00001 of 25.6 = 25.6 ÷ 100000 = .000256

.00012 of 25.6 = 12 × .000256 = .003072

These solutions (i), (ii), and (iii), are given to show the form of the written work. Many exercises should be given. There should be very many solutions of this kind in which the analysis is entirely mental.

G. The Equation with Decimals

This is a step of first importance in decimal teaching. A large proportion of the applied problems involving decimals are in the equation form. Exercises involving the equation should be given frequently for months—throughout the elementary school period. Very much work of this kind should be given for mental solutions. This is very practical. The questions on similar work with common fractions in Part II, Chapter XXVI, *F 1* and *2* should be used with each exercise.

Form of the Written Work.

(i) .4 = 2.4

.1 = $\frac{1}{4}$ of 2.4 = .6

1.0* = 10 × .6 = 6.

(ii) .32 = .72

.01 = $\frac{1}{32}$ of .72 = .0225

1.00† = 100 × .0225 = 2.25

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*This is read ten tenths.

† This is read 100 hundredths.

(iii) $1.2 = 5.4$
$.1 = \frac{1}{12}$ of $5.4 = .4\frac{1}{2} = .45$
$1.0 = 10 \times .45 = 4.5$
(iv) $3.07 = .0614$
$.01 = \frac{1}{307}$ of $.0614 = .0002$
$1.00 = 100 \times .0002 = .02$

The above analysis work should have oral-written solutions also. Every pupil should be able to go to the board with an equation and *talk* the solution as he writes it. "4 tenths is equal to 2 and 4 tenths. Then 1 tenth is equal to 1 fourth of 2 and 4 tenths or 6 tenths. 10 tenths is equal to 10 times 6 tenths which is 6. 6 is 10 tenths or the whole.

SUMMARY.

1. Operations with 10's and 100's by removals of the decimal point. (*B*, *C*, and *D*.)
   Written work.
   Mental work.
2. Operations with 10's and 100's—short methods. (*E*.)
   The rules or statements. (*E*, *4*.)
3. Partition with decimals—the analysis. (*F*.)
4. The solution of equations involving decimals. (*G*.)

Time required for the work of this chapter, including necessary reviews, fifteen days.

CHAPTER XIII

PER CENT INTRODUCED

Topics:—Interpreting the Written Language. Computing with Per Cent.

A. Aim

This section has in it nothing in the way of new decimal fraction forms or new reductions or other processes. It involves the one development—the oral and written per cent with its sign %.

The facts as to what per cent means, that 25% means $\frac{1}{4}$; 33$\frac{1}{3}$%, $\frac{1}{3}$; 75 per cent, $\frac{3}{3}$; etc.; are very easily and quickly learned. The meaning is so easily conveyed that we sometimes see it in use in second or third grades in partition work. In a grade in which decimal fractions may properly be taught, it is only a matter of minutes to teach the pupil to recognize the common equivalents.

Notwithstanding the fact that these equivalents are simple and easily memorized and applied, the language of per cent is seriously difficult. The child's troubles are due to what, to him, is a confusion of terms. He is taught, for illustration, that 25 per cent or 25% means .25. On the other hand he learns after many troubles that while .25 may be called 25 per cent, he cannot *write* it as .25% or .25 per cent without radically changing its value. This is the source of his trouble. It requires special training to teach him that when we use the written sign % or the term per cent with a number, we give the number merely a *numerator* value. 30% means that 30 is a numerator, 100 is its denominator. 75 per cent means that 75 is a numerator, $\frac{75}{100}$. $33\frac{1}{3}$ per cent means that $33\frac{1}{3}$ is a numerator, $\frac{33\frac{1}{3}}{100}$. $\frac{1}{4}$% means $\frac{\frac{1}{4}}{100}$, .75% means $\frac{.75}{100}$ or $\frac{\frac{3}{4}}{100}$.

The emphasis in the introduction to per cent, therefore, must be upon teaching the individual to think of numbers to which per cent (or %) is applied (whether in oral or in written expression) as numerators of fractions which must be given the denominator 100 expressed (in common fraction form) or understood (in decimal fraction form). The aim of this chapter is to learn to interpret and use properly the written expressions.

B. Presentation

1. The Term Per Cent

The teacher prepares a long list of numbers to be read by the pupils—to be read first as they are written and afterwards as *100ths*. The numbers should be selected with a view to making it necessary for the pupil in reading them as 100ths to reduce them, sometimes to higher and sometimes to lower denominators. They should include numbers that will necessitate the reduction of complex decimals:—

.35, .62, .3, 1.4, 2.5, 3.35, .255, .475, .9, .852, 2.00,* 3.0,* 1.4, 1.5, 2.555, .735, .02, .015, $.3\frac{1}{3}$, $.6\frac{2}{3}$, $.666\frac{2}{3}$, .625, .875, $.333\frac{1}{3}$, $.3\frac{3}{4}$, .375, .1225, .1, 1.4, $.6\frac{1}{4}$, .2, .8, $.00\frac{1}{4}$, $.00\frac{2}{3}$, $.05\frac{1}{5}$, .002, $.0033\frac{1}{3}$, .0025, .0075, .0014, 25.62, 21.2, 15.6, etc.

The teacher now gives the pupils the terms per cent. The term and the per cent sign, %, are written on the blackboard. "The term means 100ths and when we find an expression of 100ths—.25, .32, .05, .01, .10, etc.—we may read the numbers as per cents or as 100ths as we prefer. When we have a number that is not 100ths, we must change it to 100ths if we wish to read it as per cent."

The pupils are now required to read the exercises again, this time as per cents. They will read 1.4 as 140%; 2.555, as $255\frac{1}{2}$% or 255.5%; .002, as $\frac{2}{10}$% or $\frac{1}{5}$%; .0003, as $\frac{3}{100}$%; 25.62, as 2562%.

2. *Changing Common Fractions to Per Cents*

In this work the fractions selected for change to per cent language should be largely those that we wish the pupil to become thoroughly familiar with—

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* Must be read, when reading "as they are written," 2, 3. As 100ths they are read $\frac{200}{100}$, $\frac{300}{100}$.

$\frac{1}{2}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{4}{5}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{1}{20}$, $\frac{1}{25}$, $\frac{3}{20}$, $\frac{7}{25}$, $\frac{1}{40}$, $\frac{1}{50}$, $\frac{1}{200}$, $\frac{1}{400}$, $\frac{1}{300}$, $\frac{1}{500}$, $\frac{1}{150}$, $\frac{1}{250}$, etc.

The pupils should have so much of this work with halves, thirds, 4ths, 5th, 6th, 8th, 10ths, 20ths, 25ths, and 50ths, that the equivalents are memorized or so quickly reduced mentally that they are given practically without hesitation. (For the form of making the written reduction, see Chapter IX, *D*.

3. Changing Per Cents to Common Fraction Form

25%, 20%, $12\frac{1}{2}\%$, 50%, 40%, $62\frac{1}{2}\%$, $87\frac{1}{2}\%$, 75%, $37\frac{1}{2}\%$, 2%, 5%, 15%, $33\frac{1}{3}\%$, $3\frac{1}{3}\%$, 80%, 125%, 150%, 200%, 250%, 500%, 1000%, 600%, 2150%, 2500%, 3·4%, 1.25%, 125%, 2·5%, 3·3%, ·3%, ·2525%, $\cdot 6\frac{1}{4}\%$, ·2%, 2·3%, $1\frac{1}{3}\%$, etc.

These reductions, after those given in *2* above should, many of them at least, be mental or partially written. The pupil should in each reduction write the equation—$12\frac{1}{2}\% = .12\frac{1}{2} = \frac{1}{8}$, $20\% = .20 = \frac{1}{5}$, $33\frac{1}{3}\% = \cdot 33\frac{1}{3} = \frac{1}{3}$, etc. In this written work the pupil should write the per cent fractionally, with the decimal point. If, however, the fraction equivalent is not remembered the written reduction should be made in analytical form—

$$.03\tfrac{1}{3} = \frac{3\frac{1}{3}}{100} = \frac{1}{100} \text{ of } \frac{10}{3} = \frac{1}{100} \text{ of } \frac{100}{30} = \frac{1}{30}.$$
$$.2\% = .002 = \frac{2}{1000} = \frac{1}{500}.$$
$$3{\cdot}4\% = .034 = \frac{34}{1000} = \frac{17}{500}.$$
$$1\tfrac{2}{3}\% = .01\tfrac{2}{3} = 1\frac{1\frac{2}{3}}{100} = \frac{1}{100} \text{ of } \frac{5}{3} = \frac{5}{300} = \frac{1}{60}.$$

C. Computing with Per Cent

1. *Aim*

Since per cent is merely a form in which the decimal fraction 100ths is expressed when not in common fraction form, this work in computation should have a place here in connection with other decimal fraction work.

The computations used here will be the partition and the equation forms. (See Chapter XII, *F* and *G*). These are used because the two forms of work are very important in number training on account of the extent to which they enter into the common fraction and decimal fraction analyses. They serve the purposes of the outline here and at the same time there is an opportunity for further experience in them.

The aim of this section is to establish in the child's mind firmly that operations with per cent in the integral forms (15% of ——, 2.5%

of ——, .25% of ——, $1\frac{1}{4}\%$ of ——, etc.) are not possible without changing the expressions to fractional form (.15 of ——, .025 of ——, .0025 of ——, $.01\frac{1}{4}$ of ——, etc.). It is to teach him by many experiences that when in the statement of the problem to be solved the per cent is given in integral form it must be changed at once into decimal or common fraction form and the computation made with the fraction thus divided.

The exercises for partially written and mental work should be those in which the per cent is half, thirds, 4ths, 5ths, 6ths, 8ths, 10ths, etc.—those which the pupil can instantly translate into common fractions and make the computation mentally.

2. *Partition Computations—Of Work*

a. Written Work

3% of 15 =
7% of 12 =
125% of 28 =
1.5% of 20 =
300% of 14 =

.2% of 28 =
$1\frac{2}{3}\%$ of 30 =
$1.2\frac{1}{2}\%$ of 40 =
.5% of 60 =
400% of 3 =

$.33\frac{1}{3}\%$ of 8 =
$3.3\frac{1}{3}\%$ of 10 =
.8% of 6 =
$1\frac{1}{2}\%$ of 15 =
200% of 11 =

Forms of the Solutions.

1.5% of $20 = .015$ of $20 =$

$.01$ of $20 = 20 \div 100 = .2$

$.015$ of $20 = 1.5 \times .2 = .3$

$1\frac{2}{3}\%$ of $30 = .01\frac{2}{3}$ of $30 =$

$.01$ of $30 = 30 \div 100 = .3$

$.01\frac{2}{3}$ of $30 = 1\frac{2}{3} \times .3 = .5$

$.8\%$ of $6 = .008$ of $6 =$

$.01$ of $6 = 6 \div 100 = .06$

$.008$ of $6 = .8 \times .06 = .048$

$.25\%$ of $16 = .0025$ of $16 =$

$.01$ of $16 = 16 \div 100 = .16$

$.0025$ of $16 = .25 \times .16 = .04$

250% of $18 = 2.50$ of $18 =$

$.01$ of $18 = 18 \div 100 = .18$

$2.50 = 250 \times .18 = 45$

These solutions are given to show the form of the written work. It is the "Of Work" outlined in Chapter XII, *F*. The questions suggested in Part II, Chapter 26, *F*, *1* and *2* should be asked with each exercise—

How many steps in the operation?

What was the operation in the first step? (Dividing by 100, the denominator, to find .01 (one per cent)).

What was the operation in the second step? (Multiplying by ——, the numerator, to find ——.)

The pupils should have very much work of this kind. In time they should be able to ask and answer the questions fully without suggestion.

b. PARTIALLY WRITTEN WORK

25% of 12
75% of 24
$12\frac{1}{2}\%$ of 64
$37\frac{1}{2}\%$ of 48
$87\frac{1}{2}\%$ of 41
$33\frac{1}{3}\%$ of 36
$66\frac{2}{3}\%$ of 21
20% of 30
80% of 25
2% of 150

200% of 4
150% of 6
300% of 7
125% of 12
175% of 16
400% of 2
1120% of 32
160% of 15
$16\frac{2}{3}\%$ of 25
$133\frac{1}{3}\%$ of 21
$166\frac{2}{3}\%$ of 36

These exercises should be solved mentally, only the result written. The solutions are by changing the per cent to the common fraction.

Form of the Mental Work in the Solution.

25% of $12 = \frac{1}{4}$ of $12 = 3$
150% of $8 = 1\frac{1}{2} \times 8 = 12$
200% of $12 = 2 \times 12 = 24$
$112\frac{1}{2}\%$ of $24 = 1\frac{1}{8} \times 24 = 27$
175% of $21 = 1\frac{3}{4} \times 21 = 36\frac{3}{4}$
500% of $3 = 5 \times 3 = 15$

The value of such work can hardly be overestimated for training in the use of per cent. Very much of it should be done.

c. MENTAL WORK

The mental is like the "partially written" work excepting that the dictations are oral and the solutions entirely mental. Much of it should be done with 25%, 75%, 150%, 300%, $87\frac{1}{2}\%$, $62\frac{1}{2}\%$, $33\frac{1}{3}\%$, $66\frac{2}{3}\%$, etc.

The form of solution is by using the common fraction equivalent.

3. *Equation Computations*

a. WRITTEN WORK

$2.5\% = 75$
$3\frac{1}{3}\% = 18$
$1\frac{1}{4}\% = 35$
$.05\% = 45$
$.3\% = 9$
$1.5\% = 6$
$.1\frac{1}{2}\% = 18$
$.2\% = 22$
$.03\frac{1}{3}\% = 24$
$.7\% = 21$
$15\% = 18$
$.2\frac{1}{3} = 35$

Forms of the Solutions.

(i) $15\% = .15 = 18$
$.01 = \frac{1}{15}$ of $18 = 18 \div 15 = 1.2$
$1.00 = 100 \times 1.2 = 120$, the whole.

(ii) $3\frac{1}{3}\% = .03\frac{2}{3} = 18$
$.01 = \frac{1}{3\frac{1}{3}}$ of $18 = 18 \div 3\frac{1}{3} = 5.4$
$1.00 = 100 \times 5.4 = 540$, the whole.

(iii) $2.5\% = .025 = 75$
$.01 = \frac{1}{2.5}$ of $75 = 75 \div 2.5 = 30$
$1.00 = 100 \times 30 = 3000$, the whole.

(iv) $.3\% = .003 = 9$
$.01 = \frac{1}{.3}$ of $9 = 9 \div .3 = 30$
$1.00 = 100 \times 30 = 3000$, the whole.

These solutions show the forms of the written work. It is the same kind of work as that outlined in Chapter 12, *G*—the "equation work."

Questions suggested in Part II, Chapter 26, *F*, *1* and *2* should be used with every solution—

How many steps in the operation? (Two.)

What was the operation in the first step? (Dividing by ——, the numerator, to find .01 (one hundredth).)

What was the operation in the 2d step? (Multiplying by 100, the denominator, to find 1.00 (one hundred hundredths).)

This work should have very much attention.

b. Partially Written Work

In this work the teacher uses the per cents which may be handled mentally.

$25\% = 12$
$75\% = 27$
$12\frac{1}{2}\% = 32$
$87\frac{1}{2}\% = 35$
$62\frac{1}{2}\% = 25$

$33\frac{1}{3}\% = 27$
$66\frac{2}{3}\% = 19$
$37\frac{1}{2}\% = 21$
$150\% = 18$
$250\% = 37\frac{1}{2}$

$200\% = 5$
$300\% = 7\frac{1}{2}$
$1000\% = 33$
$125\% = 25$
$175\% = 14$

FORM OF THE MENTAL WORK IN THE SOLUTIONS

These solutions are based wholly upon the common fraction equivalents. The pupil translates the per cent into its common fraction equivalent and uses the latter in his mental computation.

(i) $25\% = 12$
$25\% = \frac{1}{4} = 12$
$\frac{4}{4} = 4 \times 12 = 48$, the whole.

(ii) $87\frac{1}{2}\% = 35$
$87\frac{1}{2}\% = \frac{7}{8} = 5$
$\frac{1}{8} = \frac{1}{7}$ of $35 = 5$
$\frac{8}{8} = 8 \times 5 = 40$, the whole.

(iii) 125% = 25

$125\% = \frac{5}{4} = 25$

$\frac{1}{4} = \frac{1}{5}$ of $25 = 5$

$\frac{4}{4} = 4 \times 5 = 20$, the whole.

The pupils should have much work of this kind.

c. Mental Work

This is like the partially written work excepting that the dictations are oral and the solutions mental—no written work. It is not necessary to urge the value of this work or to suggest that it be used very much.

Summary.

1. The troubles that the beginners have and how they may be remedied. (*A*.)
2. Reading decimal fractions in regular decimal form, as 100ths, in terms of per cent. (*B*, *1*.)
3. Changing common fractions to per cent form and per cent forms to common fractions. (*B*, *2* and *3*.)
4. Computing with per cent. (*C*, *1*.)
   - (*a*) Partition, "of work," with analysis.
     Written, partially written, and mental computations. (*C*, *2*.)
   - (*b*) Equations with analysis.
     Written, partially written, and mental computations. (*C*, *3*.)

CHAPTER XIV

THE EMPHASIS POINTS IN DECIMAL FRACTIONS

Topics:—Reading and Writing Decimals. Reductions and Computations. Definitions and Rules

A. Reading Decimals

1. Proper Fractions

Pupils should be able to read any decimal fraction giving such emphasis upon *numerator*, and *denominator* as will make the fraction clear to a listener.

2. Decimal Mixed Numbers

(*a*). As They Are Written

The pupil should be able to read any mixed number with proper emphasis upon the *integer*, the *and*, the *numerator*, the *denominator* so that a listener need have no question as to the number.

(*b*). As Improper Fractions

The pupil should be able to read any decimal mixed number as a mixed number or as an improper fraction as directed. This training to go from mixed number to improper fraction reading is very important indeed in decimal fraction work. He should be able to read without hesitation exercises like the following:

1.3 as $1\frac{3}{10}$ or as $\frac{13}{10}$
2.65 as $2\frac{65}{100}$ or as $\frac{265}{100}$
10.00 as 10 or as $\frac{1000}{100}$
2.50 as $2\frac{5}{10}$ or as $\frac{250}{100}$
7.0 as 7 or as $\frac{70}{10}$
17.010 as $17\frac{1}{100}$ or as $\frac{1701}{100}$

3. Using One "And." Decimal expressions require and should have but one * *and.* Pupils should be trained to recognize and observe this carefully in their reading under all conditions and in all connections. "One hundred *and* fifty" for 150 should not be tolerated. Strict attention to the proper use of *and* (by the *teacher* as well as by the pupils) contributes very materially to accuracy in reading and writing decimal expressions. It keeps attention centered upon the *integer*—the *and*—the *numerator*—the *denominator.*

B. Writing Decimals

The pupil at the close of the work in the subject should be able to write correctly any decimal mixed number or fraction from oral dictation or from dictations in which the expressions are fully written out in words. The only condition for such a test should be that the dictations be clearly given—with due voice attention in oral dictation to clearness as to the *integer*—the *and*—the *numerator*—the *denominator*.

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* There is one exception (but only an apparent exception) to this rule of the single "and"—mixed numbers having complex fractions have necessarily a second "and." $1\frac{1\frac{3}{4}}{5}$ is read *one—and—*one and three fourths—*fifths.* $2\frac{3\frac{1}{3}}{10}$ is read *two—and—*three and one third—tenths.

(*c*). REDUCTIONS

(*1*) Reducing Common Fractions to Decimal Form.

Pupils should be able to reduce any ordinary common fraction or mixed number to decimal fraction form by the short analysis process; and halves, 3ds, 4ths, 5ths, 6ths, 8ths, 20ths, 25ths, and 50ths by analysis or by inspection.

They should be able to reduce common fractions—

(i) To simple decimals, when possible.

(ii) To complex decimals *exact* to a required denomination as—

$\frac{1}{3}$ to decimal form exact to 1000ths.
$\frac{3}{7}$ to decimal form exact to 100ths.
$1\frac{5}{6}$ to decimal form exact to 10000ths.

(iii) To an approximate decimal fraction having a stated denominator, the pupil giving the nearest approximation using + or −:

$2\frac{5}{7} = 2.7143-$, or
$2\frac{1}{3} = 2.333+$

The fractions to which much attention should be given in these reductions are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{20}$, $\frac{1}{25}$, and $\frac{1}{50}$. The pupil should in time be able to give the decimal equivalents for these fractions without analyzing.

(2) Reducing Simple Decimal Fractions.

(i) To Higher Equivalents.

Writing the series of equivalents of fractions and mixed numbers—

.3, .30, .300, etc.
.75, .750, .7500, etc.
1.2, 1.20, 1.200, etc.

(ii) To Lower and Lowest Terms.

.70 to a lower equivalent = .7, the lowest term.

1.3500 to a lower equivalent, 1.350; to its lowest simple decimal equivalent, 1.35.

.020 = .02, its lowest term or lowest simple decimal equivalent.

3.0030 = 3.003, its lowest term.

Pupils should be in the habit of using fractions in their lowest terms in computations. The teacher should insist upon this use of the lowest form.

(iii) To Given Terms—Higher or Lower.

This trains to recognize the lowest term of a simple fraction.

The following illustrates what is meant by this form of work:

Give the 1st, 3d, and 5th terms of the series of simple decimal fraction equivalents of—

.050, .100, 1.01, and 2.0100
.050 = .05, .0500, .050000
.100 = .1, .100, .10000
1.01 = 1.01, 1.0100, 1.010000
2.0100 = 2.01, 2.0100, 2.010000

(3) Reducing Complex Decimals to Higher and Lower Equivalents.

The pupils should be able to make such changes by the analysis processes as outlined in Chapter X.

Very many of the reductions should after a time be made also by inspection. Exercises which involve the complex decimal fractions from halves, 3ds, 4ths, 5ths, 6ths, and 8ths, should be given so many times that the pupil will recognize the forms of the decimal equivalents. The pupil should learn to recognize the following:

.325 as $.32\frac{1}{2}$ or $.3\frac{1}{4}$
$.03\frac{1}{2}$ as .035
$1.2\frac{2}{3}$ as $1.26\frac{2}{3}$ or $1.266\frac{2}{3}$
$2.43\frac{1}{3}$ as $2.433\frac{1}{3}$ or $2.4\frac{1}{3}$
$1.4\frac{1}{5}$ as 1.42
$2.6\frac{4}{5}$ as 2.68
$1.3\frac{1}{4}$ as 1.325 or $1.32\frac{1}{2}$

$.64\frac{3}{4}$ as $.6475$ or $.647\frac{1}{2}$
$2.18\frac{1}{2}$ as $2.1812\frac{1}{2}$ or 2.18125
16.875 as $16.8\frac{3}{4}$ or $16\frac{7}{8}$
$.1625$ as $.1\frac{5}{8}$ or $.16\frac{1}{4}$ or $.162\frac{1}{2}$
$.0433\frac{1}{3}$ as $.04\frac{1}{3}$ or $.043\frac{1}{3}$
$.4375$ as $.4375$ or $.4\frac{3}{8}$
$.362\frac{1}{2}$ as $.36\frac{1}{4}$ or $.3\frac{5}{8}$

(*4*). Reducing Per cent and % Expressions to Decimal Forms.

This work is of great importance because of the wide use of these terms in practical life.

It involves the changing of common fractions to "per cent," and "%" expressions to decimal fraction and common fraction forms.

(i) Common Fractions to "%" Forms.

$\frac{1}{4} = .25$ or 25% (the two forms in which 25 per cent may be written).
$\frac{5}{8} = .62\frac{1}{2}$ or $62\frac{1}{2}\%$ or 62.5% (the three forms in which $62\frac{1}{2}$ per cent may be written).
$4\frac{1}{5} = 4.20$ or 420%
$5\frac{1}{6} = 5.16\frac{2}{3}$ or $516\frac{2}{3}\%$
$1. = 100\%$ or $1.00 = \frac{100}{100}$
$6 = 600\%$ or $6.00 = \frac{600}{100}$
$2 = 200\%$ or $2.00 = \frac{200}{100}$

(ii) Per cent Forms to Decimal and Common Fraction Forms.

$60\% = .60$ or $\frac{60}{100} = \frac{3}{5}$
$250\% = 2.50$ or $\frac{250}{100} = 2\frac{1}{2}$
$37\frac{1}{2}\% = .37\frac{1}{2}$ or $\frac{37\frac{1}{2}}{100} = \frac{75}{200}$ or $\frac{3}{8}$

$$635.2\% = 6.35\tfrac{1}{5} \text{ or } \tfrac{635\frac{1}{5}}{100} = \tfrac{3176}{500} \text{ or } \tfrac{794}{125}$$
$$.2\% = .00\tfrac{1}{5} \text{ or } \tfrac{\frac{1}{5}}{100} = \tfrac{1}{500}$$
$$.25\% = .00\tfrac{1}{4} \text{ or } \tfrac{\frac{1}{4}}{100} = \tfrac{1}{400}$$
$$3\tfrac{1}{3}\% = .03\tfrac{1}{3} \text{ or } \tfrac{3\frac{1}{3}}{100} = \tfrac{1}{30}$$
$$.87\tfrac{1}{2}\% = .00\tfrac{7}{8} \text{ or } \tfrac{\frac{7}{8}}{100} = \tfrac{7}{800}$$

D. Operations

1. Addition

Besides the addition of simple decimals and decimal approximations, the pupil should know how to add complex decimals which do not have a common denominator.

$3.4\frac{1}{3} + 6.256\frac{2}{3}$ involves the reduction of the two fractions to a common denominator—

$$\begin{array}{r} 6.433\frac{1}{3} \\ \underline{+6.256\frac{2}{3}} \end{array}$$

The pupils should be able to make any such necessary changes in complex decimals so that the addition may be effected. This is the point in the subject that requires special attention.

2. Subtraction

The subtractions of simple decimals or decimal approximations has no difficulties. In the subtraction of complex decimals which do not have a common denominator, the pupils should recognize the need of reducing one or both of the fractions.

$$\left.\begin{array}{r} 18.2\frac{2}{3} \\ \underline{-3.5875} \end{array}\right\} \text{should be} \left\{\begin{array}{r} 18.2\frac{2}{3} \\ \underline{-3.5\frac{7}{8}} \end{array}\right\} \text{or} \left\{\begin{array}{r} 18.2666\frac{2}{3} \\ \underline{-3.5875} \end{array}\right.$$

Pupils should be able to make such solutions properly.

3. Multiplication

In the multiplication of decimals the use of complex decimals is very rare in practice. If it were desired to operate with such fractions it would be like the multiplication of mixed numbers *vertical* form in common fractions.

$$\begin{array}{r} 4.2\frac{1}{3} \\ \underline{\times\ \ .4\frac{1}{5}} \end{array}$$ (See Part II, Chapter XXIV.)

4. Partition—Computations

The pupils should be able to solve by written analysis any reasonable exercise in "of work" involving either general decimals—.45 of 12 =, .3 of 14 =, 1.25 of 24 =, etc. —— or per cent forms of statement. They should be able to ask the questions upon the analysis (See Chapter XII, *F*) and to answer them without suggestion, in this way stating the number of steps and the character and purpose of each.

They should be able to make written solutions of equations in decimal fraction and in "per cent" language; also to ask and answer the questions suggested for such work (Chapter XII, *G*).

They should be able to solve such exercises in the partially written form.

They should be able also to solve mentally any of the less complicated problems in "of work" and "equations"—problems involving

25%, 75%, $33\frac{1}{3}$%, $66\frac{2}{3}$%, $12\frac{1}{2}$%, $37\frac{1}{2}$%, $62\frac{1}{2}$%, $87\frac{1}{2}$%, 125%, 150%, 200%, 500%, 1000%, etc.

These computations both the written and the mental should receive much attention because of the wide use of "of work" and "equations" in percentage later.

5. *Division*

The pupil should be able to point off correctly in the quotient in any division operation with simple decimals. The two matters to be carefully looked after in division are—

(1) This "pointing off" in the quotient; and (2) division when the *divisor* is complex.

In practical computations, quotients are either simple decimals with low denominator, or they are approximations.

When the dividend has a decimal complex fraction, the fraction is reduced to a higher term—to the limit of the desired approximate quotient.

When the *divisor* has a decimal complex fraction, the reduction of the fraction to a higher term lengthens the operation and is therefore not practical.

In $3.84\frac{2}{3} \div .4\frac{1}{6}$, to give a closely approximate quotient, the pupil should change dividend and divisor to 6ths. This was the pupil's method of dividing in common fractions. The dividend becomes 23.08 and the divisor 2.5.

In $4.562\frac{1}{3} \div .26\frac{1}{4}$, the dividend and divisor may be reduced to 12ths. This is what the pupil will suggest from past experiences.

In $5.628 \div .4\frac{1}{8}$, dividend and divisor should be reduced to 8ths. The dividend will then be 45.024, the divisor 3.3.

The pupil should be able to make divisions involving fractions in the *divisors* by reducing dividend and divisor to their lowest common denominator. He should have much experience in this line of work because of its practical value in shortening the operations.

$9.863 \div .2625$ is much shorter when changed to $9.863 \div .2\frac{5}{8}$ and then to $78.904 \div 2.1$.

E. Definitions and Rules

a. The Terms and Definitions

Per cent.
Decimal Fraction
Simple Decimal Fraction
Complex Decimal Fraction
Repeating Decimal
Circulating Decimal
Repetend

Per cent means hundredths.

Decimal Fraction.—A decimal fraction is a fraction whose denominator is one of the powers of ten. This denominator may be written or it may be understood.

Forms of Decimals.—Decimal fractions may be in either one of two forms—with the denominator understood or with the denominator written.

A simple decimal fraction is one whose numerator is integral.

A complex decimal fraction is one whose numerator is a fraction or a mixed number.

A repeating decimal fraction is one in which one or a group of the quotient figures regularly repeats.

The figure or the group of figures in a quotient that repeats is called a repetend.

b. RULES

How do we point off in multiplication of decimal fractions?

In multiplying decimal fractions we point off in the product as many fractional places as there are in both multiplier and multiplicand.

How do we point off in division of decimal fractions?

In division of decimal fractions we point off in the quotient as many fractional places as the number in the dividend exceeds the number in the divisor.

SUMMARY.

- The emphasis in reading and writing fractions and mixed numbers. (*A* and *B*.)
- The reductions that must be emphasized (*C*.)—
  - Common fractions to decimals, exact and approximate.
  - Simple decimals—
    - To higher equivalents.
    - To lower decimal fraction equivalents.
    - To given terms higher or lower.
  - Complex decimals to higher and lower equivalents with special attention to those from halves, 3ds, 4ths, etc.
  - Per cent expressions—
    - Common fractions to per cent forms.
    - Per cent forms to regular decimal forms.
- Emphasis points in Addition, Subtraction and Multiplication. (*D*.)
- Emphasis in partition on the analytical solutions in "of work" and "equations." (*D*, *4*.)
- The emphasis in division on reducing dividend and divisor to a common denominator when the *divisor* is *complex*. (*D*, *5*.)
- The definitions and rules. (*E*.)

PART IV

CUBIC MEASURE

CHAPTER I

THE TERMS AND DEVELOPMENT APPARATUS

A. The Terms Capacity and Content Discriminated

Almost necessarily, the first consideration in the development of cubic measure is the definite selection of terms to be employed so that throughout the work careful discrimination shall be made between computations of the *matter* in bodies and computations of *capacity*. A sewing thimble has in it a quantity of matter, it has also capacity. In measuring, we must discriminate. What we measure must be the one or the other definitely.

Clear imagery is as necessary in cubic measure as in any other form of number work. Such imagery, however, cannot be developed unless in our language and computations we make this discrimination sharply in every part of the work.

Our regular term *capacity* fully meets our needs as one of these special terms. It is perfectly definite at the outset, the *usual* term by which we refer to what a given space will "hold." The term *content* must be chosen to express quantity of matter as opposed to capacity. In its ordinary use it is flexible. When we say content we may be referring to the quantity of the matter in the object or to its full displacement. Our use of it here, however, must be strictly limited. We must apply it *in all cases* to quantity of *matter*. When we refer to the content of a glass tumbler we *must* mean the quantity of glass in it. The advantage in the selection of this word is that later after its use for a long time in this restricted sense, the individual will make more intelligent use of it in its *wider* meaning. If our aim were to develop *content*, we could not adopt a better plan than this of first using it in a sharply restricted sense and then later permitting it to take on its *general* meaning. It is going from percept to concept—nature's way.

Cubic measure is left out of our number training unless we provide that this sharp discrimination is made in all our developments and computations between measures which relate to content, in the limited sense, and those which relate to capacity. When we measure, we should definitely measure the one or the other. In that way we are forced to discriminate—to determine in every practical problem whether it relates to the material in the object or to what it would "hold." If we are to measure entire displacements, as in the case of measuring a hollow sphere, a tin cup, or any other object not of solid matter throughout, we should do so in set terms. The fact that such measurement is required should show in the language used, so that there could be no possibility of matter and capacity not being definitely discriminated in the case. When we wish to measure any form of *capacity*, the word *capacity* must be our sole reliance for technical expression in the case. When it is the *matter* in a body, our word is *content*. If we speak of content, we *must* have in mind the matter, the quantity of material, that is in the body or object. The content of any ink-bottle would be the quantity of glass that there is in it. If we want its *capacity*, we should so state. If we mean the total displacement, we must ask for the *content* "*as if it were solid glass*."

B. Capacity and Content Developed

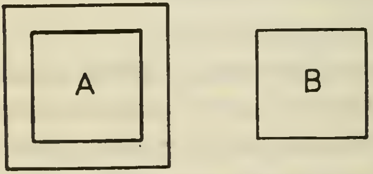
1. *Tools and Material*

a. The Measuring Cup

For the proper development of these terms the teacher should have 100 or more kindergarten cubic-inch blocks; three or four quarts of fine, dry sand free from lumps and pebbles;

and a cubic-inch cup into which one of the cubes will fit as perfectly as possible consistent with easily putting it in and taking it out. The cube should just fill the cup—the top being level with the top of the cup. There are other receptacles that will be discussed later.

This cup is to be used as a measuring cup to determine capacities and contents, the sand being the material for measuring. The cup will have to be made to order by a tin-smith. The manner of construction is to cut four pieces of tin of the proper size for the sides of the cup, these to be soldered on the outside. The piece for the bottom should be about $1\frac{3}{4}$ inches square so as to form a projecting shelf around the cup. This allows for the strong soldering of sides to bottom on the outside. By this plan of construction, the inside of the cup is free from irregularities due to inside soldering. The handle for taking the cube out of the cup may be a tack or small finishing nail driven into the cube and left to project enough to serve as a handle.



A. The cup looked at from above.

B. The cup's side view.

b. STRIKING

Definition and Method.

Besides these, there should be a straight-edge, a wooden foot-rule, perhaps, for "striking." "Strike" as used here, is a term, nearly obsolete now, that applied to the act of drawing a "straight-edge" across the top of the peck or half-bushel or bushel measure to level off the wheat or other grain that was being measured. It left the grain level with the top of the measure, pushing off the excess grain after every part had been filled. The term is introduced here for want of a better to express this act of filling out and leveling off in this way the top of a box or cup (or other receptacle whose edge is regular so that the process leaves a uniform flat surface) of sand when we wish to obtain a quantity of the latter exactly equal to the capacity of the container. The "striking" process, to give a proper surface, requires that the receptacle be slightly over-full beforehand. The best evidence that every part is filled is through there being an oversupply that is pushed off in the act of striking.

c. WHY USE SAND

The material for our development work here has been chosen with a view to its availability for use with all kinds of receptacles and because of the more satisfactory results that its use gives. Water is not practicable for use for several reasons:

(1) We can not use it in receptacles that are not water-tight.

(2) Wooden and paper boxes unless waterproofed absorb water.

(3) Attraction and, under conditions resulting from handling, repulsion by the sides of our tin cup would render its measurements so unreliable as to defeat their purposes.

(4) Measurements of content by displacement would be possible with waterproof objects only.

Sand is free from these objections, its particles are fine so that "striking" gives a surface that is loose, and "shaking down" does not materially change content.

d. The Receptacles

In the matter of the receptacles to be used in connection with capacity and content measurements, we should be governed by the rule that choice be confined to such as may be "struck" to a *level surface*. We have shown that it is impracticable to use water or other liquid. Water capacity as well as sand capacity must, however, be considered. When we ascertain with sand that the capacity of a receptacle is a certain number of cubic inches, it should be understood that its water capacity would be the same. This requirement would limit us to the use only of containers which would "strike" *level.* A common water-pitcher, for illustration, with spout above the level of a part of the top is susceptible of "striking" for sand, but practically could not be used in our work. Sand would stand in the trough of this elevated spout after "striking." Some of it would be above the level of that in other parts of the top. Water, however, would seek its level. The water capacity would be less than this sand capacity. Our aim is not by any means to measure this or that vessel. It is intended that our work shall be a means to an end—that end, the terms *capacity* and *content.* This end is served only when capacities are general—alike for liquid and dry material.

2. *The Cubic Inch*

The teacher taking one of the cubic-inch blocks—

"We call this a cubic-inch. This means that the amount of wood in the block is one cubic-inch. When a person uses that word he is telling how much."

"This is a cubic-inch cup. This means that it will hold so much of anything."

(The teacher puts the cube into the cup, takes it out, and puts it back again so that all observe how closely it fits.)

The teacher fills the cup with sand, strikes it carefully, and pours the sand on a paper, placing the cube beside it.

"Which is greater, the quantity of wood in this block or the quantity of sand here?"

("The quantities are equal.")

"How much sand?"

("A cubic-inch.")

"How much wood?" ("A cubic inch.")

Another "struck" cupful is added to the sand on the paper.

"How much sand in the pile?"

("Two cubic-inches.")

"How much greater is the quantity of sand on this paper than the quantity of wood in this cube?"

("Twice as great.")

"Show me an equal quantity in wood." (The pupil shows two of the cubes.)

"How much?"

The teacher now adds one or more cupfuls of sand to the pile.

"What quantity of sand now?"

"Show me an equal quantity in wood."

The quantity of sand is again increased, the quantity is asked for, and the pupils are required to show an equal quantity of wood.

The aim here is to fix in mind at the outset that cubic inch is a word which relates to quantity and not to something having eight corners.

3. Capacity

"This cup holds a cubic inch. We say that its capacity is a cubic-inch."

"This cup has capacity enough for one cubic-inch of sand."

"You may find the capacity of this teacup" (putting a teacup on the table).

The pupil does the work. He is required to "strike" each cupful (cubic-inch cup) that he puts into the teacup. He fills it a little overfull. It should stand on a piece of paper so as to catch any spilled sand and also the sand pushed off in striking.

"How did you find the capacity?"

("By measuring it with sand.")

The teacher shows him about filling the teacup overfull, about striking, and about measuring the sand spilled or pushed out on the paper. The word strike should be given to the pupils in connection with this work and it should be used in classwork freely. The spelling of the word must not be overlooked.

The amount of sand on the paper (number of cubic-inches) deducted from the amount used in filling the cup gives the amount in the cup. This amount measures the cup's capacity. Parts of cubic-inches in this sand work should be approximated with the eye—$\frac{1}{8}$ cubic-inch, $\frac{3}{4}$ cubic-inch, $\frac{1}{2}$ cubic-inch, $\frac{1}{4}$ cubic-inch, etc.

NOTE.—This cup should now be set aside as an auxiliary *measuring* cup for use in finding capacities of receptacles which require too much time to fill with the cubic-inch cup, one cupful at a time. If the capacity of this teacup should be found to be 8½ cubic inches, it could be used to advantage in measuring sand into a box having a capacity of 10, 20, or 30 cubic inches—great enough to use one or two or three cupfuls for filling it in greater part, the remainder being measured in with smaller cups or boxes whose capacities have been measured and with the cubic inch cup. The teacher who desires to work rapidly will lay aside several cups and boxes of various sizes that have been measured making a careful record of the capacity of each. These she may use for rapid measuring work. These must, of course, be receptacles that may be "struck."

The capacities of two or three other cups or boxes should be measured in the same way. If for this second measuring exercise a receptacle larger than the teacup is chosen, the teacup may be used as an auxiliary measuring cup to hasten the work as suggested above.

4. *Content*

The term content is now used.

"This cube has a quantity of wood in it, a cubic inch. We speak of this as its *content.* The content of these two cubes is two cubic inches. Content means quantity of material. This piece of coal" (a piece of coal about the size of an egg) "has *content*, a quantity of material in it."

"Let us find the content of this piece of coal."

The teacher selects any box or cup that can be "struck"—one large enough to hold the piece of coal so that the latter will be entirely below the top of the receptacle. The capacity is measured with the sand and recorded. The coal is now placed in the empty receptacle, sand is measured into it, and the quantity of sand noted. The pupils should suggest the next step—the difference between the capacity of the box and the amount of sand that is held with the piece of coal is the content of the coal. This difference is computed and stated.

5. Displacement

The term displacement should now be introduced to describe this process of finding content by letting it displace sand in a measured vessel. The term should be put into regular use in the class by the question after these measurements, How did you find the content? (By displacement.) In order to give the term larger use, the teacher should ask the question, How *would* you find the content of this ——? How would you find the content of these irons? Etc.

"When we say that we measure by displacement we mean that we use the article in place of sand in measuring. We find how much less sand it takes. This is the content of the article."

Another Problem.

"Here is a rock. You may find its content."

The pupils will find it. They will select a container large enough to hold the rock, find its capacity, then find the amount of sand that with the rock will fill it "strike" full, and then take the difference as the content of the rock.

"How did you find the content?" ("By displacement.")

Questions for Regular Use.

What do we mean by the content of an article?

(The quantity of material in it.)

What do we mean by capacity?

(The quantity of material that article will hold.)

What is the unit, the one, of content?

(The cubic inch.)

What is the unit, the one, of capacity?

(The capacity of the cubic inch cup.)

When I asked for the content of the piece of coal, what did I want to know?

(You wanted to know how many cupfuls there would be if it were all ground up * and measured.)

"You may find the content of this pair of scissors."

The content is found as in the case of the rock. The questions are asked.

"You may find the content of this teacup."

(This should be a cup that has not been used in the class—a fresh cup.)

This puts before the pupil for the first time a question which will compel him to show how deeply into his mind the terms capacity and content have filtered.

(The teacher may find that pupils will begin work on the *capacity* rather than the content. The only form of interference by the teacher that would be at all admissible would be the question, "What do we mean by content?" "Content." "What do we mean when we say content?" The teacher must not, however, permit any pupil to *answer* the question. It is asked for the purpose of concentrating attention, of forcing discrimination in the matter.)

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* This conception of content must be cultivated. It fits in with their imagery. The conception of the content of the piece of coal or the rock or other irregular object in terms of the *cube* is out of line with the pupil's experiences. He has, however, measured sand in the cup, and the grinding up coal or rock into the form of coal dust or sand is an ordinary sight where coal is handled or heavy wheels strike rock on paved streets or gravel roads.

The content is found just as that of the rock and the piece of coal were found. The cup is placed upright in a box deep enough to receive it entirely. The capacity of the box is found and the number of cubic inches of sand that the box, with the upright cup inside, will hold. The difference is the content of the cup—the number of cubic inches of material in the make up of the cup were it ground up and measured in the cubic-inch cup.

"How did you find the content of the cup?"

("By displacement.")

Additional Problems.

The content of this empty inkbottle.

The capacity of this saucer. Its content.

The capacity of this book.*

The content of these nails. The capacity of the nails.

The capacity of this pasteboard box. Its content.

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* This, of course, has no capacity. Questions of this character should be given occasionally in order to compel attention to questions and the habit of judging whether or not the question relates to a possibility.

SUMMARY.

The need of discriminating, in the development work in cubic measure, between content and capacity. (*A*.)

Capacity developed by measurements with sand. (*B*, *3*.)

Content developed by measuring by displacement with sand. (*B*, *4*.)

Time required for the work of this chapter, five days.

CHAPTER II

MEASURING CAPACITY

Topic:—Measuring Capacities that are in the Form of Right Rectangular Prisms

A. Measuring with Blocks

1. Aim

The aim here is to begin the development of the regular way of measuring capacities—measurement made in terms of cubic inches in a row along one side of the bottom of the box, the number of such rows in the bottom layer, and the number of layers. There is the further purpose to conduct the work in a way to lead the pupil to think all regular computations of this kind analytically—in these terms of units in a row, rows, and layers.

The work involves the use of our cubic-inch blocks and with them of boxes made of pasteboard or other material—boxes that have square corners, rectangular prisms in shape. These should, if possible, have all three dimensions in even inches (inside measurement) so that rows and layers of cubes will fit fairly well. A half dozen boxes of different sizes are all that would be needed for a class, these being enough to serve all necessary development purposes with blocks. The work later in which blocks are not used does not require the use of boxes selected with reference to special inside dimensions.

2. *The Boxes*

The teacher will have difficulty in finding boxes of reasonable size that have dimensions in even inches. A box for this purpose should not be larger than $6'' \times 10''$ or $8'' \times 10''$ and $3''$ to $4''$ deep. Larger boxes require too many blocks and serve no better purpose. To secure proper dimensions in a box it may be necessary to cut it longitudinally and perhaps transversely and telescope the *bottom* for reduction in one direction and the *sides* for reduction in the other direction, the parts being fastened together afterwards by using paste or glue. If there is manual arts work in the school, the boxes might be made to order. Small wooden boxes of thin material may be cut down to desired dimensions and the parts fastened together by outside cleats. The fact that there are ridges in the pasteboard boxes that have been telescoped, so that blocks do not fit closely wherever those ridges occur, is not a serious matter. The purpose of this use of the blocks is entirely for the development of the row and layer mode of measuring. It is true that on account of the ridges the capacities found are not accurate, but the inaccuracies are not so glaring as to take attention from the main point—that it is possible to measure capacities by filling them with cubic inch blocks which occupy the spaces fully and counting the blocks.

Two of these boxes should have *two* of the dimensions with a half inch in each—as 7″ by $6\frac{1}{2}$″ by $2\frac{1}{2}$″ deep or $8\frac{1}{2}$″ by $5\frac{1}{2}$″ by 3″ deep. These will be used to give the pupils an opportunity to think about —— blocks and a half block in a row, or —— rows and a half-row, or —— layers and a half layer, or a combination of such conditions. The pupils need not make half-blocks for such boxes. They will have no difficulty in the matter because they can see where the half-blocks and the half-rows, or the half-rows and the half-layer would be. This provides the necessary imagery for problems that have one or more fractional dimensions—the ordinary problems.

3. The Plan of Development

The teacher places one of these selected boxes before the class and asks how its capacity may be measured. The answer will be, By measuring it with sand.

"Yes, but we can also measure it by seeing how many of the cubic inch blocks it will hold."

"You may measure it in that way."

This will be a class exercise in which one or two pupils place the blocks in full view of the other members of the class.

The teacher must see that one full row of blocks is put in before a second is begun. This row may be along one side or across the end of the box, it does not matter.*

"How many blocks in the row?"

"You may now put other rows in until you complete a layer."

The computation for this layer is now placed on the blackboard under the teacher's direction—

(Assuming that the box has these dimensions.)

1 row = 8 cu. in.

6 rows = 6 × 8 cu. in. = 48 cu. in., 1 layer.

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* In all this "blocks in a row and rows" measuring, there is no choice as to whether the pupil's "row" runs the long way or the short way of the receptacle that is being measured. One pupil in making a computation may consider the rows as running the *short* way while another making the same computation may consider them as running the *long* way. The one should be as acceptable as the other. The teacher's attitude in this matter should be that it is absolutely immaterial.

Note—Base.—The teacher, after the completion of the layer, upon whichever side the pupil may choose to construct it, must introduce the term *base* as indicating the side along which the first layer is constructed. This side must be spoken of as *base* in every construction thereafter.

The term is introduced by simply telling the pupils that the side of the receptacle along which any one has built his first layer is his base. One child may consider one side as base; another child, another side.

The term is almost as necessary for class use in dictations and in pupils' oral work. If put into use now as describing the side upon which the pupil constructs his first layer in these measures of capacities, the child's vocabulary will be broadened by its use and it will give him a conception of "base" as applied to solids that will not have to be modified later.

The teacher, in questioning on a construction, will ask one pupil and then another, What did you use for the base? The reply will be by showing the "base" side. He will say, I used this side; or, I used the bottom; or, I used this end.

The pupils are now directed to put in the rest of the layers.

The computation on the blackboard, assuming 4 layers, is now completed—

1 row = 8 cu. in.

6 rows = 6 × 8 cu. in. = 48 cu. in., 1 layer on the base.

4 layers = 4 × 48 cu. in. = 192 cu. in., capacity of the box.

The above is the form in which the analytical solutions should be written.

Questions.*

(1) "How did you measure the capacity of the box?"

("With cubic inch blocks.")

(2) "How many ways have we of measuring the capacity of this box?"

("Two. With sand and with the blocks.")

(3) "How many ways have we of measuring the capacity of this cup?"

("One. With sand.")

(4) "Why may we not measure the capacity of the cup with the blocks?"

("Because the blocks will not fit into it.")

(5) "What was your base?"

Others of the prepared boxes should be measured in the same way, computing the capacities in the analytical form as given above. The questions should be asked with each.

With these prepared boxes there should be also several that are not rectangular prisms. There should be, if possible, boxes whose bases are rhomboidal, one or two with triangular bases, others with hexagonal or trapezoidal bases. The plan should be to measure one or two of the prepared boxes. In each the blocks will fit satisfactorily and the computations will be made regularly. The teacher without remark now presents for measurement one of the oblique angled receptacles. When the pupil tries to fit the blocks into the box in this case he will be unsuccessful in making even the *first row*. The *first block* of the first row will not fit into its corner. The teacher's part in the work of putting in the blocks is to insist that they must fit into the box so as to fill *all the space*. If the shape of the box is such that the blocks can not be made to fill the whole space the teacher and the pupil must agree that "this box can not be measured with blocks." In this case, this conclusion is drawn because the blocks do not fit into the box. The question should be asked at once, If we can not measure it with blocks, how can we measure it? The answer to this is, of course, that the only way to measure it is with the sand. This question should be asked whenever measurement with blocks is found impossible.

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* The teacher should use these questions regularly. The 4th question, particularly, is important because we shall make use of it later to bring out as strongly as possible the fact that with blocks or by the use of the foot-rule we are able to measure vessels that have corners that cubical blocks will fit into—vessels in the form of right rectangular prisms. The teacher should seek for opportunities to ask the questions.

Others of the prepared boxes are now measured and then one or two of the boxes with oblique angled bottoms.

In connection with this work the instruction in corners and solid angles (see *5* below) should be given in order to make use of the language to express why blocks will or will not fit into a given receptacle.

The questions that have been suggested, particularly the question, How many ways have we of measuring this box? asked in connection with the measurements of each of the prepared boxes, should have brought out that—

(1) The boxes that may be measured with the blocks may be measured in two ways.

(2) The boxes that may not be measured with the blocks may be measured in one way only.

4. Angles—Right and Oblique, Surfaces—Plane and Curved

It is assumed that pupils of the age to take up the study of cubic measure have been taught right, acute, and obtuse angles. Besides being able to recognize and name such angles or to make them from oral dictation—make an obtuse angle, make an acute angle, make a larger acute angle, make a larger obtuse angle, make a smaller obtuse angle, make a right angle, etc.—the pupils should be able to classify these angles as right and *oblique.* When the teacher points to or makes an angle, the pupil should describe it as an oblique angle, acute; or, as an oblique angle, obtuse; or, as a right angle. In speaking of *corners* and *solid angles* in this cubic measure work, it is very desirable to make use of the term *oblique* as the single opposite of *right* or *square;* hence the introduction and establishment of *oblique.*

Pupils should understand and be able to use the terms plane surface and curved surface. What is a plane surface? A plane surface is one that is flat.

5. *Corners and Solid Angles*

The cubic inch blocks have *corners*, solid corners. These corners are square because the angles on the sides or faces which meet at these corners are right angles. Other square, solid corners should be found. Door and window casings, picture frames, remnants of lumber from carpenter shops, etc. furnish models of this class of corners. Oblique corners—oblique because one or more of the angles of the sides or faces which meet there are acute or obtuse—may be found sometimes on door or window casings, picture frames, geometrical solids, lumber remnants, etc.

There are four square, *solid angles* in the bottom of our tin cubic inch cup. A square corner will fit closely into a square, solid angle. Oblique, solid angles are illustrated by the solid angles in receptacles whose bases or ends are triangular, rhomboidal, or hexagonal, etc. Of the sides or faces which meet to form an oblique, solid angle, one at least has an oblique angle at the solid angle.

This language of corners, square corners, oblique corners, solid angles, square solid angles, and oblique solid angles should be used in connection with the measuring of capacities by means of the blocks.

The following questions suggest points that must be fully developed before passing to "Measuring Without Blocks." (See *B* below.)

Why can we measure this with blocks?

(Because it has four solid angles that are square, and its faces are plane surfaces. The blocks will fit into it.)

Why can we not measure the capacity of this box with blocks?

(Because its solid angles are oblique. The blocks will not fit into oblique solid angles.)

What kind of receptacles can we measure with blocks?

(We can measure with blocks, only those receptacles that have square solid angles and plane surfaces.)

These questions must be asked regularly in connection with the work of measuring *with blocks*, because the pupil must depend upon what he learns here to determine when working later *without blocks* whether the latter would or would not fit into a given receptacle. Before he takes up "Measuring Without Blocks" he must be able ordinarily to state from the appearance of the capacity to be measured whether cubical blocks would fit into its corners and whether or not the faces are plane. Blocks would not fit into a box that does not have all its faces plane and all its solid angles square.

B. Measuring Without Blocks

1. Using the Prepared Boxes

The same boxes should be used again as follows:

"In how many ways can we measure the capacity of this box?"

("With sand and with the blocks.")

"How do you know that the blocks would fit into it?"

("Because all the solid angles are square and the faces are all plain. The blocks would fit into it.")

"Is there any way to find the capacity of this box without using sand or blocks?"

(One of the prepared boxes is used here.)

"Is there any way of finding the number of cubic inch blocks in a row without actually putting in the blocks?"

There should be the proper response from one or more pupils otherwise the suggestion should be made that "this foot rule might tell us."

The width or the length (as a pupil may choose) is measured and from this the pupil states how many of the cubes would fit in to make a row.

"How did you find out?"

("I measured with the rule. There would be one block for every inch. —— inches, therefore —— cubic inches in a row.")

"Is there a way to find how many rows in the first layer?"

No suggestion will be needed here. The pupil measures and replies.

The question of how he found out is asked.

("—— inches, therefore —— rows in a layer.")

The question as to the number of layers is asked in the same way, followed by the question as to how he found out.

The computation is now made. (We will

assume that the dimensions of the box were 10″ by 9″ by 5″ deep.)

9 in. wide ∴ 9 cu. in. in 1 row.

10 in. long ∴ 10 rows.

10 rows = 10 × 9 cu. in. = 90 cu. in., 1 layer on the base.

5 in. deep ∴ 5 layers.

5 layers = 5 × 90 cu. in. = 450 cu. in., capacity.

"How did you find the capacity of this box without sand or blocks?"

("I found the number of cubic inch blocks that there would be in a row, the number rows, and the number of layers by measuring with a rule.")

Another of the prepared boxes is now discussed as to how many ways of measuring and how we know that blocks would fit into it. (Its solid angles are square and its faces plane.) It is measured and the computation arranged in form as shown above.

A box with oblique solid angles is now taken up. The question is asked, Would the blocks fit into this? Why would blocks not fit into it? Because it has solid angles that are oblique and blocks fit only into boxes with square solid angles. The question is now asked, How many ways have we of measuring its capacity? One, with sand.

In this way the prepared boxes and the

oblique angled boxes are taken up, discussed as to ways of measuring, and the computations when made arranged in analytical form as shown above.

(These discussions and computations bring out that the capacities of cups, saucers, bowls, lard-cans, playroom buckets, baking powder cans, can covers, dinner-bells, inkbottles, etc., and all receptacles that have oblique solid angles may be measured in one way only—with the sand; and that the capacities of vessels that have square solid angles and plane faces can be measured in three ways—with the sand, with the blocks, and by using the foot-rule to find out what the blocks would tell us were we to use them.)

The two prepared boxes that have a half inch in one or two of their dimensions should be taken up last, before passing to the "miscellaneous exercises" that follow. They have put the blocks into them, so far as possible without half-blocks, and have computed their capacities. They should now measure them again without the blocks, finding blocks in a row, rows in a layer, etc. with the foot-rule.

2. *All Kinds of Receptacles—Without Blocks*

The teacher should now take up the measurement of the capacities of all forms of receptacles *without blocks*. The pupils have learned that there are three ways in which such measurements may be made—

(1) With the sand. This form may be used in *all* cases.

(2) With the cubical blocks. This form may be used only with receptacles which have all their faces plane and all their solid angles square.

(3) With the footrule without the blocks. The footrule tells us how many blocks there would be in a row, how many rows in a layer, and how many layers. We can measure with the footrule in this way only capacities having all faces plane and all solid angles square.

a. THE FIRST STEP IN A SOLUTION

By using all forms of capacities for measuring exercises now, the pupil is given experience in stating from the appearance of a receptacle whether it may be measured with the sand only or with the footrule and the blocks and why. The pupil's *first step* in the solution of every problem must be to ask and answer the two questions as to measuring the capacity with blocks—

Can I measure this with the blocks? (He must answer and give the reason as above.)

How many ways have I of measuring it?

(He must answer whether he could measure it with sand only or in the three ways.)

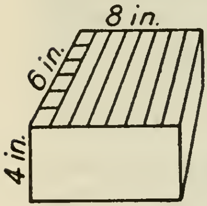
The work of measuring the receptacles that have plane surfaces and square solid angles with the computations written out in proper analytical form, (see *1* above) completes the development of the measurement of capacities. Exercises for practice should be given almost daily for many weeks.

b. DIAGRAMS

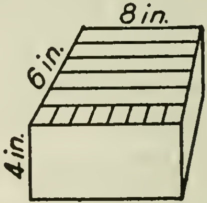
The diagram will be very valuable for use after the pupil begins to measure receptacles without objects. It will be a means by which the pupil may give full expression to his mental picture of the solution operation. The diagrams should be developed during the objective period because it will be a picture of the measurement with blocks.

After the measurement of a box with blocks or through the use of the footrule, the pupils should be required to draw a diagram of the "base" of the box showing *one* full row of the blocks as constructed, and the remaining rows in outline. These diagrams should be drawn with care as to neatness but not necessarily to scale. If a diagram shows the general arrangement of the blocks in the pupil's first row and of the rows of his first layer it serves fully the purpose desired. It would be a very simple matter to teach the pupil to make the diagram to show also an outline of the box. This would greatly aid imagery and expression. This outline need not be drawn to scale.

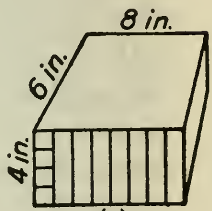
The following diagrams show how one form of picture could be made to show any plan of solution. In (1) the rows have 6 blocks; in (2), 8 blocks; in 3, 4 blocks. Diagrams should be required in a large proportion of the solutions. In many problems the diagram of the *base* only should be asked for.



(1.)



(2.)



(3.)

c. Kinds of Fractions

In measuring dimensions of boxes, it is suggested that as this subject relates to modes of computing capacity, the use of difficult fractions would very greatly reduce the number of box measurements that could be made within a lesson period and thereby hinder the chief object in view—the development of cubical measurements. Work with difficult fractions should be postponed until a later period in our cubic measure program. The teacher is wise who, at this time, sees to it that the fraction in any dimension is not higher than eighths. The teacher should suggest after a measurement for one of the dimensions of a box that shows a fraction like $\frac{1}{16}$ of an inch, We will call it $\frac{1}{8}$; or, if it shows $\frac{13}{16}$ of an inch, We will call it $\frac{3}{4}$; or, if $\frac{10}{16}$ or $\frac{9}{16}$ of an inch, We will call it $\frac{1}{2}$. This would make for rapid computations and give time for a greater number of exercises. Later, computations should be made with measurements correct regardless of the fractions.

d. FORM OF ANALYTICAL COMPUTATIONS

The following solution of the problem of the capacity of a box 16 in. long, 12 in. wide, and 10 in. deep shows the form or arrangement that the teacher should require in all solutions:

16 in. long ∴ 16 cu. in. in 1 row.

12 in. wide ∴ 12 rows.

12 rows = 12 × 16 cu. in. = 192 cu. in., 1 layer on the base.

10 in. deep ∴ 10 layers.

10 layers = 10 × 192 cu. in. = 1920 cu. in., capacity.

3. Problems Without Objects

Problems in capacity computations in regular form are now given—objects (receptacles) not being used, the teacher giving the length, breadth, and depth of the imaginary receptacles to be measured.

Find the capacity of a box 14 in. long, 12½ in. wide, and 5 in. deep. The box has plane surfaces and all of its solid angles are square.

There is a box having plane surfaces and square solid angles. It is 10 in. long, 9½ in. wide, and 8¼ in. deep. Find the capacity.

Many of these should be given. The computations should be written out analytically. Diagrams of the *bases* should be required for many of the solutions. The two questions suggested in *2 a* should not be overlooked in any solution.

C. Problems from Diagrams of Bases

1. Forms of the Problems

Many of the exercises in capacity measurements without objects should be dictated by the teacher by means of diagrams of the "bases" with a statement in each case of the depth or the height. The aim is to enable the teacher to make use in this way of problems in which the measurements can be made by blocks or the foot-rule but which at the same time are a departure from regular right rectangular prism forms. The following diagram is illustrative of this class of exercises:

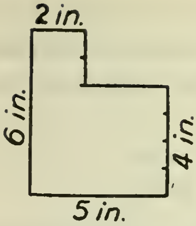
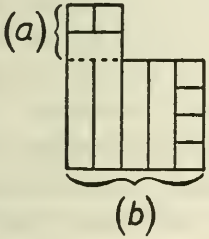


Diagram of the base of a receptacle.
Depth, 3 in.

2. *Measuring the First Layer*

The arrangement of the first layer in uniform rows on such a base is impossible without first separating the base into rectangular parts, and a computation is impossible without such uniform rows. If we try to arrange rows horizontally on this base, the first 4 rows will each consist of 5 cubic inches, the next 2 having 2 cubic inches each. If we try to arrange them vertically, 2 rows will have 6 cubic inches each and 3 will have 4 each. Solution, therefore, without an adjustment is not possible. The following diagrams show the changes that might be made so that under

1.



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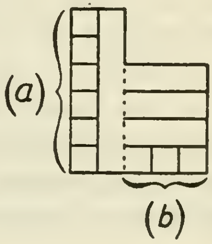

either there would be two parts of the base each with its set of uniform cubic inch rows from which the number of cubic inches in the full layer may be computed.

3. Importance of the Work

There is nothing in exercises in measuring rectangular prism forms to impress upon the pupil the special importance of *rows* and *uniform rows* of units in the first layer. There is nothing in such constructions to break the rows, nothing to call for effort in arranging them. He may make them but it is because we have told him to do so. These irregular bases, however, present *problems* of uniform rows of blocks. The pupil is compelled to find ways of forming rows or sets of rows. This makes him understand the suggestion, and even to anticipate it, that next in importance to the question, Can this be measured with blocks? is the question, Can the first layer be arranged in rows of units, and how?

In practical life, problems of this kind are common and are the only ones relating to the measurement of rectangular prisms that are not of the most elementary character. The real training of the pupil in cubic measurement applications, therefore, hardly begins until he takes hold of this class of problems, until he faces solutions in which he is forced to provide for the separation of bases into two or more rectangular parts in order that he may arrange his first layer in as many sets of rows of measuring units.

4. *Form of the Analytical Solutions*

In the analytical solution of problems of this class the pupil should arrange the work to show the computation of each part of the first layer and then the computation of the capacity of all the layers. The solution of the problem in 2 above from diagram (2) would be somewhat as follows:

(*a*) 6 in. long ∴ 6 cubic in. in 1 row.
2 in. wide ∴ 2 rows.
2 rows = 2 × 6 cu. in. = 12 cu. in.
(*b*) 3 in. wide ∴ 3 cu. in. in 1 row.
4 in. long ∴ 4 rows.
4 rows = 4 × 3 cu. in. = 12 cu. in.
24 cu. in. in a layer.
3 in. deep ∴ 3 layers.
3 layers = 3 × 24 cu. in. = 72 cu. in., capacity.

5. *Forms to be Measured*

Diagrams for solution should have various forms but all solid angles must be square—of a shape to admit measurement by blocks.

The following forms will suggest the forms of capacities to be measured.

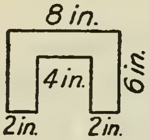

Diagram of base of prism. Depth, 4 in. Find capacity.

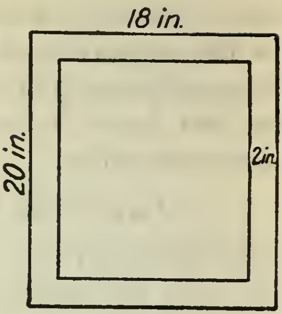

Diagram of excavation. Depth of excavation, 6½ in. Find capacity.

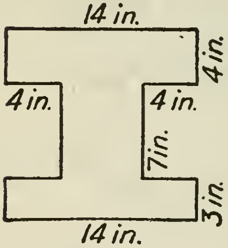

Diagram of excavation 4 in. deep. Find capacity.

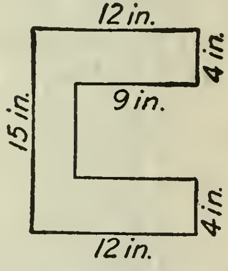

Diagram of excavation 5½ in. deep. Find capacity.

6. Questions

In every solution but particularly in every one in which a subdivision of the "base" is necessary, the pupil should ask and answer some or all of the following questions:

(1) Can this be measured with blocks or the footrule?

(2) Why? (Because its solid angles are square and its faces are plane surfaces.)

(3) Can the first layer of blocks be arranged in rows without making the base into parts?

(4) Into how many parts must the base be made?

SUMMARY.

The development of capacity measurements—units in a row, rows of units in a layer, and layers—with cubic-inch blocks. (*A*.)

Solid corners and solid angles, square and oblique. (*A*, *4* and *5*.)

Measuring with the footrule in place of the blocks. (*B*.)

Measuring all forms of receptacles—the pupil to choose between sand measurements and block or footrule measurements. (*B*, *2*.)

The value of diagrams. (*B*, *2*.)

Measuring irregular receptacles that have square solid angles—from diagrams. (*C*.)

Time required for the work of this chapter, including reviews, fifteen days.

CHAPTER III

MEASURING CONTENT

Topics:—The "solid" terms. Measuring the content of rectangular prisms.

A. Aim

The pupil's experience in content measuring has been confined to measurements by displacement. He has learned to measure capacity by using the sand cup and, with receptacles that have only square solid angles, by means of blocks or with the footrule.

The aim of the work in this chapter is to teach to measure right rectangular prisms with blocks and then with the footrule.

The measurement of capacities and contents in other than right rectangular prisms is a later step.

B. Material

The teacher should provide herself with several models of cardboard or wood in the form of rectangular prisms. These models should be two inches or more in thickness so as to have at least two "layers" but they should not be more than 10 to 12 inches in length. Those which are to be used for the introductory work should have length, breadth, and thickness in even inches. This is because we have no half cubic inch blocks. No block should have a content of more than 100 cu. in. because our supply of cubes for constructions is limited. For the work of computations with the footrule, but without the cubes, there may be models with dimensions not in even inches and having contents of over 100 cubic inches. Besides the right rectangular prism models, there should be several models having oblique solid corners—rhomboidal prisms, right triangular prisms, right hexagonal prisms, etc.

C. Terms

There are certain terms (some of them introduced in Chapter II) that it will be necessary to use in connection with this content work. Some of them relate to surfaces—plane surface, square, rectangle, parallelogram, triangle, rhombus, rhomboid, pentagon, hexagon and circle. It is assumed that these were carefully developed in connection with surface measure work and are now in the pupil's working vocabulary. There should be no end of the reviews of these forms.

1. "Surface" Terms

Plane Surface.—A plane surface is a flat surface.

Polygon.—A polygon is a plane surface having straight sides.

Rectangle.—A rectangle is a plane surface having four straight sides and four square corners

Square.—A square is a rectangle having equal sides.

Parallelogram.—A parallelogram is a plane surface having four straight sides, the opposite sides being equal and parallel.

Triangle.—A triangle is a plane surface having three straight sides and three corners.

Rhombus.—A rhombus is a parallelogram whose angles are oblique and whose sides are equal.

Rhomboid.—A rhomboid is a parallelogram whose angles are oblique and having two sides longer than the other two sides.

Pentagon.—A pentagon is a plane surface having five straight sides.

Hexagon.—A hexagon is a plane surface having six straight sides.

Pupils should be able to recognize these forms and to define them.

Some of these terms are used to describe prisms. In such descriptions they apply to the "bases." Pupils should learn them for use not only for surface descriptions but for use later for naming "solid" forms.

- Surface Outline
  - Triangle
  - Quadrilateral
  - Quadrangle
    - Parallelogram
      - Rectangular
        - Oblong
        - Square
      - Rhomboidal
    - Trapezoid
    - Trapezium
  - Pentagon
  - Hexagon
  - Octagon

2. *The "Solid" Terms*

a. WHEN INTRODUCED

The terms prism, triangular prism, etc. should be introduced after the computation work in capacity has been begun. The development plan in cubic measure is based upon measurements of specific solids—solids that the teacher places before the class or describes in diagrams of the "bases." The pupil examines the form to be measured and determines with the eye or by actual trial whether the measuring blocks will fit into it. The term prism or rectangular prism or cylinder does not enter into the thought in making the decision.

The name of the form to be measured is therefore not necessary to the computation work, a knowledge of the name of the form would not contribute to the measurement.

On the other hand, in the introduction of the terms, there must be very close observation of the form to which the term in each case applies. The computation work provides for such observation in the very best objective way. In determining with the blocks, or with the eye later, whether he could measure a given form for content or capacity, he must examine it as to square solid angles or corners and as to the forms and relative positions of the faces. To decide that he can measure it with blocks, he must see in it clearly every condition that enters into the definition of the form.

The proper place in the development order for the introduction of the terms, therefore, is after the beginning of the work of computation.

b. The Terms to be Developed

In selecting the cubic measure or "solid" form terms to be taught we should include—

(1) Those that the pupil has learned to measure.

(2) The names and descriptions of the additional forms that should be a part of everyone's stock of general information.

The following is suggested as the minimum list of such solids and "solid" terms:

- Solids
  - Regular solids
    - Prisms
      - Triangular
      - Quadrangular
        - Rectangular
          - Cube
          - Oblong
        - Rhomboidal
      - Pentagonal
      - Hexagonal
      - Octagonal
    - Cylinders
  - Tapering Solids...
    - Pyramids
    - Cones
- Solid angles*
  - Square
  - Oblique
- Solid corners*
  - Square
  - Oblique

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* This outline, on account of the age of the pupils to whom cubic measure is given, limits the term

c. The Terms Defined

Prism.—A regular solid whose bases are equal parallel polygons and whose other faces are parallelograms.

Triangular Prism.—A regular solid whose bases are equal parallel triangles.

Quadrangular Prism.—A regular solid whose bases are equal parallel polygons having four sides.

Rectangular Prism.—A regular solid whose bases are equal parallel rectangles.

Cube.—A rectangular prism having equal faces.

Rhomboidal Prism.—A regular solid whose bases are equal parallel rhombuses or rhomboids.

Pentagonal Prism.—A regular solid whose bases are equal parallel pentagons.

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solid angle to an interior corner of a receptacle—a corner formed by three or more sides meeting at a point. Such an angle is technically a trihedral or polyhedral angle. The *exterior* corners of prisms—the corners formed by three or more sides meeting at a point—are described as solid corners. This gives us a simple, definite, expressive language without a misuse of terms. They were introduced and defined in connection with capacity measurements in Chapter II, *A 5*. They should be reviewed frequently with objects which illustrate them.

Hexagonal Prism.—A regular solid whose bases are equal parallel hexagons.

Octagonal Prism.—A regular solid whose bases are equal parallel octagons.

Cylinder.—A regular solid whose bases are equal parallel circles.

Solid.—Anything that has content or capacity or both.

Note.—The term solid should be very carefully developed so that there will be no doubt about what, in the pupil's mind, it signifies. A solid may have content or capacity or both.

The following is given to suggest the method of development:

This block is a solid because it has content. This piece of coal is a solid. This rock is a solid. The interior of this ink bottle is a solid because it has capacity. The interior of this empty box is a solid. The interior of this empty milk bottle is a solid. The glass part of the ink bottle, the glass part of the milk bottle, the wood part of this box are solids. The ink bottle, including its interior, and the empty milk bottle, including its interior, are solids having both content and capacity. After the terms prism, rectangular prism, cylinder, etc. have been taught, the forms of the various content and capacity solids should be asked for. A form should not be called a prism or a cylinder unless it fully answers the definition. The interior of the milk bottle is in part cylinder; in part its form has no name.

Regular Solid.—A solid in which the size and shape remain the same throughout its length, from one base to the other. It does not taper.

Pyramid.—A solid whose base is a polygon and whose other faces are triangles which meet at a point.

Cone.—A solid whose base is a circle and which tapers uniformly from the base to a point.

d. "PRISM" DEVELOPED

The term prism is first developed by the use of all prism forms. Geometrical forms may be available for use. Small pieces of lumber remnants may usually be obtained. Lead pencils, rubber erasers, may be used as illustrations of prisms. The enterprising teacher will find many objects to illustrate one and another of the prism forms—rectangular, triangular, hexagonal, etc.

In studying a form to determine whether it is or is not a prism, the following questions show what the requirements are that are involved in the term:

As to the bases—

Which surfaces shall we call bases? (Surfaces that are equal to each other and parallel to each other.)

Are these two surfaces *equal?*

Are they *parallel?*

Are they *plane* surfaces?

What is a plane surface?

Now as to the other surfaces of the object—

Is this surface a *plane* surface?

What is a plane surface?

Is it a parallelogram? Why? (Pupil makes this proof through the definition.)

(These two questions are asked as to each of the other lateral faces.)

Definition.—A prism is a regular solid whose bases are equal parallel plane surfaces and whose sides are parallelograms. The term regular must be introduced. In every prism used in this connection, the questions must be asked. Is it regular? Why?

For this development work on the term *prism* the teacher must make use of *all* prism forms. The separation of these prisms into classes—rectangular, triangular, hexagonal, etc.—*must not be attempted* until the pupils are able to recognize the prism in *whatever form* it may be presented. Then it will be proper to discuss the forms again and classify them as rectangular, triangular, etc. Such classification will be found very simple if deferred until the general term prism is thus *learned.*

e. PRISMS CLASSIFIED

The child being able to recognize the prism in any of its forms, the teacher takes up with the objects the classification of prisms according to the forms of the bases. Whenever a pupil names an object as a rectangular or triangular or other kind of prism, the question, Why? should be asked. The answer is the definition, Because its bases are—

The teacher takes up the object, the pupil points out (See *D* below) and names the bases (rectangle or triangle or hexagon, etc.) and the prism is named accordingly:

Rectangular prism—because its bases are rectangles—

A rectangular prism is a regular solid whose bases are equal parallel rectangles.

What kind of solid corners?

(Square solid corners.)

Triangular prism—because its bases are triangles—

A triangular prism is a regular solid whose bases are equal parallel triangles.

What kind of corners?

(Oblique solid corners.)

Pentagonal prism—because its bases are pentagons—

A pentagonal prism is a regular solid whose bases are equal parallel pentagons.

What kind of corners?

(Oblique solid corners.)

The forms for the description of hexagonal prism are similar:

What kind of corners?

(Oblique solid corners.)

The *questions* suggested above for prisms in general should be asked in the discussion of each prism when classifying as rectangular, triangular, etc. A prism that the pupil calls hexagonal must meet all the requirements of the questions. The matter of the corners is important. We shall find that we can construct with the blocks only prisms with square solid corners.

D. How to Define Objectively

In learning the definitions of terms and in giving them in reviews, the teacher should require the pupil to give each with the object in his hand. He should be required to point to each part as he refers to it in the definition. For illustration: The pupil is defining triangular prism—

A triangular prism is a regular solid (showing it and drawing his hand over it from end to end to show its regularity) whose bases (putting the prism between his hands with his palms on the bases) are equal (with motions toward the bases indicating equality in surface) parallel (with motions showing parallel directions of the bases) triangles (tracing with a finger the three sides of one of the bases and touching each of its three corners) and whose sides (running his palm over each side in succession) are parallelograms (with finger showing the parallel direction of the opposite edges of each of the sides one after the other).

This will serve to show the form in which definitions should be required. The child who can define each form of solid thus objectively could not be accused of not knowing what his definitions mean.

E. Content Measurements

1. Measuring with Objects—Cubes

The teacher selects a rectangular prism model and asks the pupils to *build* a similar form with the cubic inch blocks. We will suppose that this model is 8″ by 6″ and 4″ thick. The first step in the measuring development is to go over the questions to prove—

(1) That the block is a prism.

(2) That it is a rectangular prism.

The definitions of *prism* and rectangular prism must be required.

The work of construction is now taken up. This work must be so directed that the pupil constructs *one* row—an 8-inch or a 6-inch row as he may choose—then the other rows of a full layer, and then the other layers. There are questions which should be used in connection with this construction in order to direct the child's thought into the line to be followed in the computation later—

How many blocks in this row?
How many rows in this layer?
How many layers?
What is the form of the block?
(Rectangular prism.)
Is it regular? Why?
What kind of corners has it?
(Square solid corners.)

The computation of the content * is now taken up—

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* The teacher must be very careful in this matter of *content* measure to be specific and consistent in the use of the term. If, for illustration, she were to ask for the content of a solid rubber ball, there would be no misunderstanding as to what the content includes. If, however, she were to ask for the content of a soft rubber ball (hollow, filled with air) the real content would be impossible without cutting the ball open so as to fill the inside with sand to get the actual displacement due to the actual *rubber* in the ball. If the content of such a ball is desired *as if it were solid rubber*, the teacher *must* so specify. A pasteboard prism 6″ by 4″ by 2″ would have an actual content equal to what the amount of paper in it ground fine would prove to be when measured in the cubic inch cup. If such a problem were given to the child without any statement further than to find the content, the measurement *must be* by displacement and the actual amount of the paper must

1 row = 6 cu. in.

8 rows = 8 × 6 cu. in. = 48 cu. in., 1 layer at the base.

4 layers = 4 × 48 cu. in. = 192 cu. in., content.

In how many ways may we measure it?

(In two ways—by displacement with sand, and with the blocks.)

Another rectangular prism is selected for measurement. The questions suggested to be asked by way of proving that it is a prism and that it is a rectangular prism and regular are asked as before. The definitions of prism and rectangular prism are required. The form is now copied with the blocks as before. The questions about the number of blocks in a row, the rows in a layer, and the number of layers, and the questions as to the form and corners, are again asked. The computation of the content is written out in proper analytical form.

In how many ways may we measure this block?

(In two ways—with sand by displacement and by constructing the form with the cubes.)

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be found. The box would have to be opened to make the measurement. If the teacher's intention is to find the content *as if made solid throughout*, she *must* so specify. The child must not acquire the habit of using the word loosely.

The teacher now selects an oblique or other prism not rectangular for construction in the same way.

The pupil tries to build it, perhaps. The construction fails, however, because cubes with square, solid corners will not make solids with oblique solid corners. The measurement with the blocks fails, therefore.

Questions.

There are three of the questions, however, that must be used—

What kind of form in this block?

(—— prism.)

What kind of corners has it?

(Oblique solid corners.)

Is it regular?

How many ways are there to measure it?

(By displacement.)

The teacher selects another and then another model for construction, discussion, and measurement as before. This should be a judicious alternation of rectangular and other forms of prisms. The questions must be asked. From this work the pupil will soon have to recognize that with rectangular prisms the forms may be made with the cubes and the content of each thus measured. He will observe too that *only rectangular prisms* may be so measured. The computations must be written out in proper form for each exercise.

2. *Measuring Without Cubes*

a. WITH FOOTRULE

The pupils now take up the measurements without the cubes but with the footrule to find out with it what the cubes would show. This being practically a repetition of similar work with capacities the pupils will need no suggestions except as to oral work and the form of the analytical computations.

When a solid is given for measurement, the pupil should state first that it may or may not be duplicated with the cubes.

Why may the form be constructed with cubes?

(Because it is a prism with square corners.)

What do we mean by prism?

(The pupil defines it.)

What kind of a prism is it?

(A rectangular prism.)

What is a rectangular prism?

(The pupil defines it.)

In the same way (in case it is an oblique prism)—

Why may not the form be constructed?

(Because it is a prism with oblique solid corners.)

What kind of a prism is it?

(The pupil states the class—triangular, quadrangular, or pentagonal, etc.)

What is a —— prism?

(The pupil defines.)

What kind of prisms can we measure with cubes?

(Rectangular prisms having square, solid corners.)

How must we measure other kinds of prisms?

(By displacement.)

b. Form of Computation

Let us suppose that the solid to be measured involves the problem of finding the content of a right rectangular prism whose base is 8″ by 6″ and whose length is 10″.

8 in. long ∴ 8 cu. in. in 1 row.

6 in. wide ∴ 6 rows in 1 layer.

6 rows = 6 × 8 cu. in. = 48 cu. in., 1 layer.

10 in. long ∴ 10 layers.

10 layers = 10 × 48 cu. in. = 480 cu. in., content.

Many solids should be given for measurement. Some of these should involve the use of fractions. Some should be given in diagram form using the solid forms suggested for capacity work in Chapter II, *C 5*.

c. Problems Without Objects

The problems are given in the usual way—the dimensions of solids are given and the form of the solid. Diagrams of bases should be used freely.

Find the content of a rectangular prism 15 in. long, 5 in. wide, and 4½ in. thick.

What is the content of a rectangular prism 7½ in. long, 6½ in. wide, and 5½ in. thick? Many problems in diagram form like those used in capacity measurements in Chapter II, *C* 5, should be given.

F. The Cubic Foot

1. The Development

The pupils are told that besides the cubic inch there is the cubic foot. This cubic foot has 12 cubic inches in one row, 12 rows, and 12 layers.

This should now be constructed. It can not be fully constructed because of lack of inch cubes. The pupil, however, may construct 1 row, the end cubes of each of the next 10 rows, and the full 12th row. This will serve as an outline of the first layer. The corners of the other layers may be built up showing what the height of the completed construction would be. The child has constructed so much with the cubes that this will suffice for imagery purposes. The pupils should be asked to build up a cubic foot in this way frequently. The computation must now be made—

1 row = 12 cu. in.

12 rows = 12 × 12 cu. in. = 144 cu. in., 1 layer.

12 layers = 12 × 144 cu. in. = 1728 cu. in., content.

What is the form of the cubic foot?

(Rectangular prism.)

What kind of corners has it?

(Square solid corners.)

Note.—Models for School Use. There ought to be several cubic foot models available for use in the school room with cubic measure. They may easily be made of stiff pasteboard cut to proper size and put together with tough paper and paste. The child should have cubic inch and cubic foot imagery. These may be developed by the presence of models in the schoolroom and their occasional use. It is the cubic foot before the child every day for a considerable period of time that gives the child power to use the cubic foot intelligently in *estimating* content and capacity.

2. *Problems Involving Cubic Feet*

Exercises should now be given in finding capacity and content in cubic feet. These should involve fractions. The computations should be written out in full analytical form showing cu. ft. in a row, number of rows in a layer, etc.—

(1) Capacities of specific boxes having plane surfaces and square solid angles. The pupils should find the dimensions by measurements.

(2) The content of specific solids, right rectangular prisms. The dimensions should be found by the pupils.

(3) Content and capacity exercises in which the problem states the dimensions. In these, the pupil's first step should be to draw a diagram of the "base."

(4) Content and capacity exercises in computing irregular rectangular prism forms from "base" *diagrams* as in Chapter II, *C 5*. These problems should relate to walls of structures, to excavations for foundation walls and those for other uses, to walls with cross-walls inclosing areas, etc. These are exercises of the greatest value because the forms of the "solids" to be measured as to content or capacity are such as to compel the pupil to look very closely to the separation of the "bases" of his problems into rectangles so that in his arrangements of rows and layers he will not be making one part of a content or capacity overlap another.

G. The Cubic Yard

1. The Development

In the same way, the pupils are told that there is a cubic yard. This cubic yard has 3 cubic feet in 1 row, three rows, and 3 layers. The computations are made—

1 row = 3 cu. ft.
3 rows = 3 × 3 cu. ft. = 9 cu. ft., 1 layer.
3 layers = 3 × 9 cu. ft. = 27 cu. ft., content.
What is the form of the cubic yard?
What kind of corners has it?

NOTE.—The schoolroom should have its cubic yard for imagery purposes. It is a very simple matter to make a framework of wooden slats, just the outer framework for each of the 6 faces. These are firmly nailed together and the whole covered with firm paper. The cubic foot and the cubic yard have large places in practical life in building and excavating. The child's imagery of these units should be carefully developed. The pupils should be required *frequently* to build up a cubic yard with cubic feet or with inch cubes considered as cubic feet.

2. *Problems Involving Cubic Yard*

Exercises should be given in capacity and in content involving computations in cubic yards. Some of these should be in measuring specific receptacles and solids, the pupils finding the dimensions with the footrule or yardstick. Some should be given in diagram form and should include forms like those used in Chapter II, *C 5*.

Other exercises should be given in which the problem simply states the dimensions. These exercises should involve fractions. The pupil's first step in one of these solutions should be to draw a diagram of the "base" marking on it the dimensions.

H. The Cord

This should be taught at this time.

A cord of wood has 128 cubic feet. The cord form may be constructed with cubic inch blocks. Each block is called 1 ft. long. The cord may then be built up—

8 cu. ft. in a row, 4 rows in a layer, and 4 layers.

The computation is made—

1 row = 8 cu. ft.

4 rows = 4 × 8 cu. ft. = 32 cu. ft.

4 layers = 4 × 32 cu. ft. = 128 cu. ft., content of the cord.

The cord should be built up in this way frequently, the pupil each time being required to state that his cube (inch) is one foot long, one foot wide, and one foot thick.

Problems in cord measure should be given in which the dimensions are *not* in regular cord form (8 by 4 by 4).

I. Bushels and Gallons

Pupils should be taught the number of cubic inches that constitute a gallon and the number that constitute a bushel. How many cupfuls of water (cubic inch cup) would make a gallon? How many cupfuls of wheat would it take to fill a bushel measure?

1 bushel = 2150.4 cu. in.

1 gallon = 231 cu. in.

These numbers have very little meaning to a child unless he uses them many times in a constructive way. He should be required to measure out bushels, pecks, dry quarts, dry pints, gallons, liquid quarts, and liquid pints with the cubic inch cup using the sand. He should not be encouraged to memorize the number of cubic inches in any of the subdivisions of the bushel or gallon. He should find these each time by computation.

As was suggested under displacement measuring in Chapter I, *B 3* Note, the actual use of the cubic inch cup for each cubic inch of sand is entirely unnecessary. Several receptacles, large and small, that may be "struck" are measured as to their capacities and that of each is recorded. To measure out a bushel of sand the pupil takes one of these measured receptacles, a large one, and uses it until the amount measured out is nearly up to 2150.4 cu. in. The work is completed with the smaller receptacles until the right amount is reached.

Asked to measure out a peck, the pupil finds $\frac{1}{4}$ of 2150.4 cu. in. (537.6 cu. in.) and measures out the amount.

Asked to measure out a dry quart he finds $\frac{1}{32}$ of 2150.4 cu. in. (67.2 cu. in.) and measures it out.

Asked to show a liquid quart, he finds $\frac{1}{4}$ of 231 cu. in. ($57\frac{3}{4}$ cu. in.) and measures it out.

In the same way he measures out liquid pints and dry pints and half bushels.

Building Up Quarts.

The fact that there is a difference between a dry quart and a liquid quart and between dry and liquid pints is an important item in the pupil's stock of knowledge. He ought to know that his peanut pint and his milk pint are quite different measures. He should remember that the dry quart or pint is larger, considerably larger, than the liquid unit of the same name.

These facts are brought out in the measuring work suggested above. They may be emphasized in another way by building up the quarts with the cubic inch blocks.

In building up a liquid quart, $57\frac{3}{4}$ cu. in.—

3 cu. in. in 1 row.

3 rows in a layer.

6 layers.

4 additional cubes toward a 7th layer.

This construction shows $\frac{1}{4}$ cu. in. more than 1 liquid quart.

In building up a dry quart, 67.2 cu. in.—

3 cu. in. in 1 row.

3 rows in a layer.

7 layers.

4 additional cubes toward an 8th layer.

This construction shows $\frac{1}{5}$ cu. in. less than a dry quart.

The pupil should be required to remember only the gallon capacity (231 cu. in.) and the bushel capacity (2150.4 cu. in.). He should be encouraged to depend upon computation for the capacities of the subdivisions of gallons and bushels.

a. The Uses of the Various Units

In connection with the work of this chapter the teacher should bring up frequently the question of how to choose between cubic inch, cubic foot, and cubic yard in a practical problem. In measuring the capacity of this room or of the next room or of the hallway people use cubic feet. When, however, they are computing the amount of dirt to be removed in excavating the cellar or for foundation walls they talk in cubic yards. In small "solids" they use the cubic inch. The aim is to instruct the pupil in the uses of the three cubic measure units. He should understand that in *all* dry-measure and liquid-measure computations cubic inches are used.

b. The Term Content

After the development work has proceeded as far as the measurement of the "content" of rectangular prisms the teacher will begin to use problems from arithmetic textbooks. The pupil will find that these books do not limit "content" to quantity of solid matter in a body. He will find capacity problems stated as *content*. It will be necessary for the teacher to explain that the content is not necessarily limited, but may without error be applied to capacity. He should be told that the word has been given to him as relating to the measurement of *material* only, because we want him in every case to *think capacity* whenever it is *capacity* that is being computed.

SUMMARY.

- The "surface" terms that are used in describing solids with their definitions (*C*.)
- The common "solid" terms with the definition of each. (*C*, *2*.)
- The development of "prism" and the classification of prisms. (*C* *2*, *d* and *e*.)
- The development of content measurements:
  - Measuring with the blocks. (*E*.)
  - Measuring without the blocks—with the footrule. (*E*, *2*.)
- The cubic foot, the cubic yard, the gallon, the bushel, the cord. (*F*, *3*.)

Time required for the work of this chapter with the reviews, fifteen days. Chapter IV is for older classes—pupils of an upper grammar grade.

CHAPTER IV

OBLIQUE-ANGLED PRISMS AND THE CYLINDER

Topics:—Measuring Right Prisms that have Oblique Corners or Angles. Measuring the Cylinder

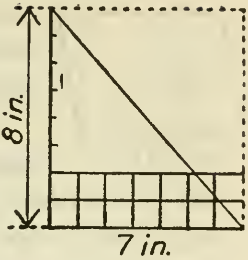
A. Measuring Prisms

It is very necessary that the pupil should learn to measure oblique-angled right prisms because the measurement of the cylinder is based upon them.

1. Right-angled Triangular Prisms

It would not be a difficult matter for the teacher to secure from a wood working shop two equal right-triangular prisms. The pupil from his experience with triangles in surface measure, suggests, when he sees the two prisms put together into the form of a rectangular prism, that he can measure the latter and that the number of cubic units in one of the two triangular prisms is equal to half of the content of the rectangular prism. He will proceed with his footrule to make the computation. Many exercises in measuring such prisms should be given, the dictations giving the bases in diagram form with statements of the lengths.

Find the capacity of a right-triangular vessel



10 inches deep whose base is as shown on the following diagram:

This diagram shows the first two of the rows.

Can we measure the base layer with blocks?

(No, not without changing it to a rectangle its double in size.)

The pupil makes the change, completing the rectangle as in this diagram.

(7 in. long ∴ 7 cu. in. in 1 row of the rectangle.)

(8 in. wide ∴ 8 rows in the first layer.)

(8×7 cu. in. =56 cu. in., the capacity of the first layer.)

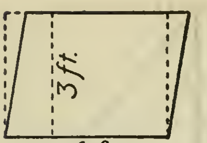
(10 in. long ∴ 10 layers.)

(10×56 cu. in. =560 cu. in., the capacity of the rectangular prism.)

($\frac{1}{2}$ of 560 cu. in. =280 cu. in., capacity of the triangular vessel.)

2. *Rhomboidal Prisms*

It is necessary to take up with diagrams of the bases the measurement of the rhomboidal prisms. It enters into the measurement of circles through the measurement of the triangles into which a circle is subdivided when we compute the cubic units on the base of a cylinder.

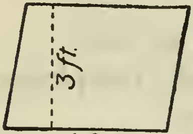


(a) 4 ft.

Find the content of a rhomboidal prism 8 ft. long having a base as in diagram (a).

(Can I measure the base layer with blocks?

No, not without changing the rhomboid to an equivalent rectangle.)



(b) 4 ft.

The change is now made, as in diagram (b).

The equivalent rectangle—

(4 ft. long ∴ 4 cu. ft. in 1 row of the rectangle.)

(3 ft. wide ∴ 3 rows in 1 layer of the rectangle.)

(3 × 4 cu. ft. = 12 cu. ft. = 1 layer on the base.)

(The rectangle and rhomboid are equivalents ∴ 12 cu. ft. on the rhomboid base.)

(8 ft. long ∴ 8 layers.)

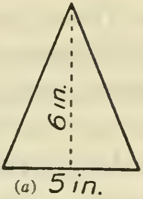
(8 × 12 cu. ft. = 96 cu. ft. = content of the prism.)

3. *Oblique-angled Triangular Prisms*

In the same way, other forms of right triangular prisms should be given to the pupils for measurement. These should be given in diagram form.

Find the content of a triangular prism 9 in. long with base as in diagram (a).

(Can I measure the base layer with blocks?)

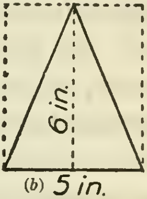


(No, not without changing it to a rectangle its double in size.)

The pupil makes the change as in diagram (b)

If required to do so, the pupil should show the rows on the diagram.

(5 in. long ∴ 5 cu. in. in one row on the rectangle.)



(6 in. wide ∴ 6 rows in the first layer.)

(6×5 cu. in. = 30 cu. in., content of the 1st layer of the rectangular prism.)

($\frac{1}{2}$ of 30 cu. in. = 15 cu. in., content of the first layer of the triangular prism.)

(9 in. long ∴ 9 layers.)

(9×15 cu. in. = 135 cu. in., content of the prism.)

B. Cylinder Measurement

1. *The Cylinder a Regular Solid*

A regular solid is one in which the size and shape remain the same throughout its length from one base to the other. A prism of whatever class keeps its form, does not taper. It is therefore regular. A cylinder is a regular solid for the same reason. A prism-like or a cylinder-like solid that tapers ever so little is out of the prism or the cylinder class at once.

These facts are practically stated in the definitions of these solids. The lateral faces of prisms could not be parallelograms under any other conditions. The teacher, however, in order to train the child to be alert and discriminating in the matter of these forms must place before the class frustums of pyramids and frustums of cones to give the members an opportunity to compare them with the prism and the cylinder. The child in looking for examples of these *regular* solids in objects outside the school-room ought to discover that tree-trunks wooden telegraph poles, and some monument shafts taper slightly and are on that account *not* cylinders or prisms.

2. *The Measurement Plan*

All changes in bases for measurement purposes have been to rectangles. The prisms with rhombus, or rhomboid, or right triangle bases were made into rectangular prisms. These we could measure with blocks. The oblique triangular prism became rhomboidal, then rectangular. As a rectangular prism we could measure it. The triangle layer is equivalent to half that of the rectangle.

The bases of the cylinder are circles. The pupil doubtless learned in his surface measure work that a circle is separable into triangles, that the triangles from any circle may be made into a rhomboid having a base equal in length to the semicircumference and a width equal to half the diameter.

The plan of work with a cylinder, therefore, is to diagram each half of the circle into a chain of triangles, to put these two chains together forming a rhomboid, and to change this rhomboid into an equivalent rectangle. (See diagrams on p. 730.) In this way the cylinder is exchanged for an equivalent rectangular prism. The measurement of the rectangle base gives the content or capacity of the first layer on the circle base of the cylinder.

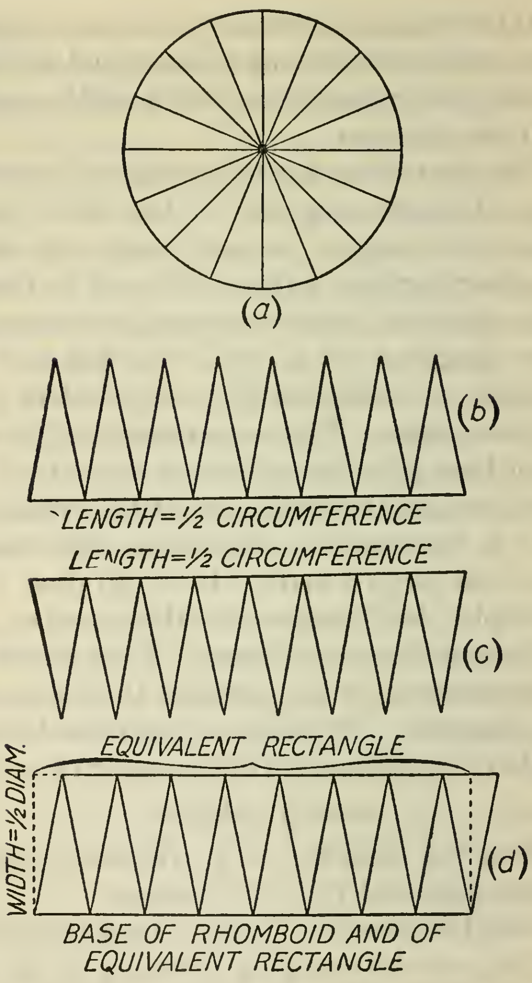
It is necessary in connection with this to teach the pupil 3.1416. He is told that, if we multiply the diameter by this number, the product is the circumference. If we divide the circumference by the number, the quotient is the diameter. To define it, 3.1416 is the ratio of the circumference to the diameter of a circle.

3. Form of Solution

Find the capacity of a cylindrical can 25 inches deep and 12 in. in diameter.

(Can I measure the "base" layer with blocks?)

(No, only by changing (pointing to the diagram) the circle base to triangles, the triangles to an equivalent rhomboid, and the rhomboid to an equivalent rectangle.



Explanations of the diagrams:—

(*a*) Diagram of circular base, 12 in. in diameter. (*b*) Chain of triangles formed by opening out half the circle. (*c*) Chain of triangles formed by opening out the other half circle. (*d*) The rhomboid and, within the dotted lines, the equivalent rectangle formed by combining (*b*) and (*c*).

(12 in. = diameter of (a).)

(3.1416 × 12 in. = 37.6992 in., circumference.)

($\frac{1}{2}$ of 37.6992 in. = 18.8496 in., length of rhomboid (d) and of the equivalent rectangle.)

(18.8496 in. = the length ∴ 18.8496 cu. in. in 1 row.)

($\frac{1}{2}$ diameter = 6 in. = width = number of rows in first layer on the rectangle.)

(6 × 18.8496 cu.in. = 113.0976 cu. in., capacity of the first layer on the rectangle and on the circle base.)

(25 in. deep ∴ 25 layers.)

(25 × 113.0976 cu. in. = 2827.44 cu. in., capacity of the can.)

The teacher should give several exercises in cylinder measurements. The pupil should make a diagram (See p. 730) in each case showing how the base is changed in measuring.

4. Reviews of Prisms and Cylinders

Much experience should be given in finding content and capacity of objects or receptacles that the pupil must measure to find the dimensions. There is more real training from finding one content or capacity in that way than from several exercises in which the dimensions required for computation are not ascertained by the pupil for himself.

C. Cones and Pyramids

There is no development for the computations of content or capacity in these forms. The fact that the content of a solid of either of these forms is $\frac{1}{3}$ the content of a cylinder or prism having the same base and height must be given to the child. He should have opportunities to verify this approximately. This is best done by displacement with the sand. To verify the fact as to the cone, the content of the cone is found by displacement. The pupil is then given a cylinder having the same diameter and the same height. He finds the content of this by displacement. The content of the latter will prove to be approximately 3 times that of the cone.

SUMMARY.

The measurement of the right triangular prism.

The measurement of the rhomboidal prism.

Measuring the oblique-angled triangular prism.

Measuring the cylinder.

Measuring the pyramid and the cone.

The work outlined in this chapter may properly be taken up in either of the last two years of the grammar school.

CHAPTER V

THE EMPHASIS-POINTS IN CUBIC MEASURE

A. As to Forms and Computations

1. The Names of the Solids

The pupil should be able to recognize the cylinder and the common prism forms—rectangular, triangular, hexagonal, etc.

2. The Forms Measured

(*a*) He should be able to state from the shape of a solid that he can or can not measure it and why.

(*b*) He must know that if the solid is "regular" and has equal parallel plane bases, he should be able to measure it if he can find the content or capacity of the first layer.

(*c*) He should know definitely that he can measure "regular" solids other than rectangular prisms only when he is able to change the bases to rectangular form.

(*d*) He has learned how to change in this way only triangles, rhombs, and circles. He should have his attention called to the fact that these are all of the forms of bases that he has learned to change to rectangular form. He can change no others.

3. Operations Stated

He should be able to state the operations in cubic measurements—

(*a*) How do we measure rectangular prisms?

("Multiply the length by the breadth and that product by the height.")

"What does that mean?"

("It means to multiply the number of cubic units on the base by the number representing the height.")

(*b*) How do we measure a right-angled triangular prism?

("Enlarge the triangular prism to a rectangular prism. Multiply the length by the breadth and that product by the number representing the height for the measurement of the rectangular prism. Half of this is the measurement of the triangular prism.")

(*c*) How do we measure a rhomboidal prism?

("Change the rhomboidal to an equivalent rectangular prism and measure the latter.")

(*d*) How do we measure an oblique-angled triangular prism?

("Enlarge the triangular to a rhomboidal prism, change the rhomboidal to an equivalent rectangular prism, and measure the latter.")

(*e*) How do we measure a cylinder?

("Change the circle base to triangles, the triangles to an equivalent rhomboid, the rhomboid to an equivalent rectangle. This is the base of an equivalent rectangular prism. Measure this prism. It gives the measurement of the cylinder.")

(*f*) Given the diameter of a circle how do we find the circumference?

("Multiply the diameter by 3.1416.")

(*g*) Given the circumference of a circle how do we find the diameter?

("Divide the circumference by 3.1416.")

4. The pupil should memorize the capacities of gallon and bushel, and the content of the cord.

5. He should be able to put capacity measurements into gallon and bushel units.

B. The Short Method

The pupil after he has learned to measure all the "solid" forms outlined for him should be given a short form of work. The short form must not be used, however, to the exclusion of the longer form. In a short method in any line of computation in arithmetic, the pupil should be required to give orally *full* analytical statements with each part of the short form. Such an analysis given with the numerical statements of the problem tends to protect the pupil against error in process. The short method in cubic measure is an arrangement of factors and divisors above and below a line for canceling.

(1) Capacity in gallons of a box $3\frac{1}{2}$ ft. long, $2\frac{1}{2}$ ft. wide, and 2 ft. deep?

The full short statement—

$$\frac{2 \times 5 \times 7 \times 1728 \text{ cu. in.}}{2 \times 2 \times 231 \text{ cu. in.}} = 130.91 \therefore 130.91 \text{ gal.}$$

The child's oral work—

"This is to be in gallons, therefore I must use cubic inches."

"$3\frac{1}{2}$ ft. long ∴ $\frac{7}{2} \times 1728$ cu. in. in 1 row."

(He writes the $\frac{7}{2} \times 1728$ cu. in. in the computation statement as he says this.)

"$2\frac{1}{2}$ ft. wide ∴ $2\frac{1}{2}$ ($\frac{5}{2}$) rows in the first layer."

(As he says this he writes the $\frac{5}{2}$ into the statement.)

"2 ft. deep ∴ 2 layers."

(He writes 2 into the statement.)

"This is the capacity in cubic inches."

"231 cu. in. in 1 gallon ∴ I change to gallons by dividing by 231 cu. in."

"There are 130.91 groups of 231 cu. in. ∴ 130.91 gallons."

(2) Capacity of a cylinder 8 ft. in diameter and 25 ft. deep?

The statement—$\frac{25 \times 4 \times 3.1416 \times 8 \text{ cu. ft.}}{2} =$ 1256.64 cu. ft.

In his oral analysis the pupil states—

"$\frac{3.1416 \times 8}{2}$ = half of the circumference ∴ the length of the rhomboid, the number of cu. ft. in 1 row of the equivalent rectangle."

"4 ft. = the radius, the width of the rectangle ∴ the number of rows in the first layer."

"The depth is 25 ft. ∴ 25 layers."

(He writes each part of the computation statement as he makes its oral statement.)

(3) Capacity in bushels of a cylinder 6 ft. in diameter and 10 ft. deep.

The statement—

$$\frac{10 \times 3 \times 3.1416 \times 6 \times 1728 \text{ cu. in.}}{2 \times 2 \times 2150.4 \text{ cu. in.}} = 227.205 \therefore 227.205 \text{ bu.}$$

The child's oral analysis states—

(i) The operations for the circumference—3.1416 × 6 ft.

(ii) The semi-circumference and with it the number of cu. ft. in 1 row of the rectangle—

$$\frac{3.1416 \times 6 \text{ cu. ft.}}{2}$$

(iii) The measurement of the first layer in cu. ft.— $\frac{3 \times 3.1416 \times 6 \text{ cu. ft.}}{2}$

(iv) The total capacity in cu. in.—

$$\frac{10 \times 3 \times 3.1416 \times 6 \times 1728 \text{ cu. in.}}{2}$$

(v) The final full statement of the capacity in bushels, as above.

The suggestions in this chapter apply to grammar school classes which have taken the work in Chapter IV.

PART V

PERCENTAGE

CHAPTER I

THE ORAL, WRITTEN, AND DIAGRAM LANGUAGE OF PERCENTAGE

Topic:—The Development of the Oral, Written, and Diagram Language "Part and Whole" Relations

A. Limitations and General Teaching Plan

1. Nature

This subject took its name originally from the way in which its fractions were expressed orally. The term applies to and includes properly only the subjects in which the measurements are entirely of the relations of parts to wholes. To be consistent in its use, it should not be made to include any of the forms of interest—simple or compound interest, bank or true discount—because in these subjects the computations do not involve considerations of part and whole relations.

The problems of percentage have to do entirely, therefore, in one form or another—

(*a*) With the part that one quantity or each of two or more quantities is of a definite whole; or,

(*b*) With the ratio that one quantity or each of two or more quantities bears to another quantity which must be regarded as a whole it being the basis of the measurement in a ratio—an implied whole and part relation.

In (*a*) there is a whole and one or more quantities that are strictly parts of that whole. For example: A bought a box for 4¢ and sold it for 3¢. The 1¢ loss is a part of the whole 4¢. He, in fact, lost $\frac{1}{4}$ of his invested capital. For a second example: A collection agency collected an account of $5 and was given $1 for the service. The $1 was a *part* of the collection, the *whole* being the $5.00.

In (*b*), there must be a *whole* and the one or more other quantities must be regarded as *parts* because the measurement of a relation (a ratio) makes it necessary that one term be regarded as a whole and the others as parts of that whole. For example: A bought a box for 4¢ and sold it for 5¢. In this case, the 1¢, the profit, is not strictly a part, but no measurement of the profit is possible unless we regard and use the 4¢ as a whole and the 1¢ as a part of that whole.

These limitations of the term being accepted, percentage should be defined as that department of arithmetic in which are grouped the subjects under which the computations have to do entirely with part and whole relations.

2. *The Development Plan*

a. AIMS

Some of the aims that must be in view in outlining a teaching plan in this subject must be the development of certain lines of power and mental habit—

(1) Power to recognize the problem as having one or more *parts* and a *whole.*

(2) Power to know definitely which of the elements of the problem are parts and which is the whole.

(3) Ability to give clearly the reason *why* the quantity which is designated as the whole is the whole.

(4) The habit of classifying the elements of the problem at the very outset of the solution as whole and parts; and

(5) The habit of giving attention, in writing the fraction that is to express the measure of the part or the ratio, to the fact that the *whole* must be its denominator—that no other quantity is admissible, under any circumstances, as such denominator.

b. The Development Language—Common Fractions

The first step, naturally, in teaching percentage is to train the child in spoken, written, and picture language of whole and part.

NOTE.—In the solutions of problems in "part and whole" (the term under which properly this subject of percentage should be known), we may use common fractions—$\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{3}{8}$, etc.; or, we may use decimal fractions in hundredths exclusively under the name "per cent"; or, we may use the common fraction forms through the solutions and translate the results, if fractional, into per cent form. The natural, simple form of solutions, the form in which a relation of part to whole first seeks expression when discovered, however, is in that of a *common* fraction. This is because the common fraction needs no explanation—the measure of this relation being in its *simplest form* in such language. We do not have to interpret or translate $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, etc. in terms of other fractions. Each tells its full story clearly. The child understands it. It is the language in which he frames fraction thoughts.

On the other hand, 50%, 25%, $66\frac{2}{3}$%, $37\frac{1}{2}$%, etc. are invariably translated mentally into $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{8}$, etc. This thinking $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, etc. when the terms 50 per cent, $33\frac{1}{3}$ per cent, 40 per cent, etc. are used is unavoidable even by the mature person, no matter how much experience he may have had in percentage operations in per cent form.

We must, therefore, separate the development of the subject into three development steps—

1st. The introduction, and the expression of part and whole relations in common fraction language;

2d. The application of this language in practical computations; and

3d. The translation of such expressions and computations into the artificial language of per cent.

This gives the mind in each step of the development but a single problem upon which to concentrate.

B. "Part Of" and "Ratio To" Developed

1. Aims

To develop the power to recognize wholes and parts.

To teach to express the measure of the relations of parts to wholes.

To teach that in "part of" exercises the *whole* is always the *denominator* of the fraction; the *part*, the *numerator*.

2. Part Of

a. ORAL DICTATIONS

Pupils respond orally and are then asked to write the fractions given. The first dictation is given abruptly without preliminary development or explanations—

"3 is what part of 4?"

The pupils will respond, "Three-fourths."

Some one is then asked to write it on the blackboard, ($\frac{3}{4}$).

QUESTIONS:—The following questions should be asked upon this and upon each of the dictations in part of, ratio to, and following steps:

"Which is the whole? Which is the part?"

("Four is the whole." "Three is the part.")

The pupils get these by inference. The "part of" in the question leads the child directly to the 4 as a whole. No teaching or explaining will be necessary.

"When you wrote the fraction where did you write the whole?"

("In the denominator.")

"Where did you write the part?"

("In the numerator.")

"What tells me the measure of the part?"

("The $\frac{3}{4}$.")

(This question should occasionally be in the form, "What tells me the value of the part?")

2 is what part of 3?

This is dictated in the same way, pupils respond orally ("Two-thirds"), and some one writes the fraction.

The questions are asked developing that 3 is the whole, 2 is the part, $\frac{2}{3}$ is the measure of the part, the whole is written as the denominator, and the part as the numerator.

Other similar exercises follow—

"4 is what part of 6?"
"3 is what part of 7?"
"5 is what part of 10?"
"2 is what part of 12?"
"7 is what part of 8?"
"3 is what part of 9?"

Many others should be given in rapid succession.

Exercises which yield improper fractions—

"5 is what part of 4?"
"8 is what part of 6?"

"7 is what part of 3?"
"9 is what part of 5?"

No explanations will be needed with such exercises.

The questions must not be overlooked.

Mixed exercises—

"8 is what part of 10?"
"7 is what part of 4?"
"10 is what part of 6?"
"9 is what part of 12?"

Many of these mixed exercises should be given.

b. Written Dictations—Oral Responses

This is the reverse of the work in (*a*). The teacher writes the fraction measure or value of one, then another, and then another "part of" expression on the blackboard for the pupils to frame the expression from which it came. Some of these fractions should be improper fractions.

$\frac{4}{5}$.

"Make a 'part of' question from this."
("Four is what part of five?")
"What is the part?"
("Four.")
"What is the whole?"
("Five.")
"Where was the whole written?"

("As the denominator.")

"Where was the part written?"

"What is the value of the part?"

("Four-fifths.")

Other dictations should be written one by one for pupils to state the "part of" question which each answers—

$\frac{2}{7}$, $\frac{3}{6}$, $\frac{5}{4}$, $\frac{7}{12}$, $\frac{12}{8}$, $\frac{9}{4}$, $\frac{15}{2}$, $\frac{2}{15}$, $\frac{8}{9}$, $\frac{9}{8}$, $\frac{14}{3}$, $\frac{3}{14}$, etc.

The questions should be asked in each case

3. Ratio To

a. Oral Dictations

These exercises are introduced abruptly in the same way as in "Part Of" above, the pupils being left to infer the "*measures*" and the *wholes* and *parts* as in that step.

(1st) Oral Ratios.

The pupils' responses are oral. A pupil writes each measure or value on the blackboard after it has been stated.

"What is the ratio of 2 to 5?"

("Two fifths.")

Questions.—The questions suggested in "Part Of" must be asked with each exercise here. (See 2 *a.*)

(2d) The Written Ratio.

The aim here is to teach the pupils to write a ratio expression. They are simply *told* how to write it.

"I asked you the ratio of 2 to 5. This is the way to write it—2 : 5."

(3d) Reading the Ratio Expression.

After writing the ratio as above (2 : 5), the teacher shows how it must be read, "The * ratio of 2 to 5."

OTHER DICTATION WORK

The teacher now gives many other ratio dictations in the same way, following each with the questions, the written form of the ratio, and the reading of the written ratio— The ratio of —— to ——.

What is the ratio of 4 to 6? What is the ratio of 5 to 9? What is the ratio of 7 to 12? What is the ratio of 7 to 4? What is the ratio of 8 to 3? The ratio of 4 to 16? The ratio of 16 to 4? The ratio of 10 to 2? etc.

b. WRITTEN DICTATIONS—RESPONSES ORAL AND WRITTEN

This is the reverse of the work in (*a*). The dictations are the *fractions*, written. The teacher writes the fraction. The pupils ask the ratio question from this fraction, (What is the ratio of, etc.?), then write the ratio in ratio language, and then read it in proper form—The ratio of —— to ——.

---

* This is the proper reading and this form must be followed absolutely in reading ratio expressions. 9 : 2 must be read, "The ratio of 9 to 2." The reading, "9 is to 2," may be used when proportion is taught but not in this development work in percentage.

The questions as to which is the part, which the whole, etc., with each exercise, are not to be omitted.

$\frac{5}{9}$.

"Ask a ratio question from this."
("What is the ratio of 5 to 9?")
"Write the ratio."
("5 : 9.")
"Read the ratio expression."
("The ratio of 5 to 9.")

The questions are now asked.

Similar work is done with other fractions—
$\frac{3}{5}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{8}{4}$, $\frac{4}{8}$, $\frac{7}{3}$, $\frac{3}{7}$, $\frac{4}{12}$, $\frac{12}{4}$, etc.

4. *Part Of and Ratio To—Mixed Exercises*

These, as the title indicates, are alternate "part of" and "ratio to" exercises aiming to train the pupil to pass from the one to the other, also to read any given fraction at one time as a "part of" and at another time as a "ratio to" expression. The questions suggested, the ratio readings, etc. are to be used here as in exercises above. These exercises should be as varied as possible, the teacher going from "part of" to "ratio to" and the reverse in all practicable ways so as to bring out the fact that in whole, part, value, whole as denominator, and part as numerator, the two forms of expression are alike.

"8 is what part of 12?"

"What is the whole?" "The part?" "The measure?"

$\frac{8}{12}$.

"Write this fraction as a ratio expression."

("8 : 12")

"Read it."

("The ratio of 8 to 12.")

"The whole?" "The part?" "The denominator?" "The numerator?"

$\frac{8}{7}$.

"This is a 'part of' fraction." "Ask the 'part of' question."

"Write a 'ratio' to expression from it."

("8 : 7")

"Read the 'ratio to' expression."

"What, in the fraction, is the part?" "The whole?"

9 : 8.

"Read it."

("The ratio of, etc.")

"Ask a 'part of' question having the same value."

("9 is what part of 8?")

"The part in the 'ratio to' expression?"

"The whole?" "The numerator?" "The denominator?" "The value?"

"The part, whole, numerator, the denominator, the *value* in the 'part of' expression?"

5. Summaries and Reviews in Part Of and Ratio To

The aim of these exercises is to center the attention upon the following facts relating to the fractions which express the values of "part of" and "ratio to" expressions—facts that are fundamental in percentage:

(*a*) That the whole is the denominator.

(*b*) That the denominator is a whole.

(*c*) That no number other than the whole may be used as a denominator.

(*d*) That a part is always a numerator.

(*e*) That a numerator is always a part.

Where, in the fraction, do we put the whole?

Where is the part?

In the fraction which expresses the value—

What is the denominator? (The whole.)

What is the numerator? (The part.)

If a number is a part where must we write it? (In the numerator.)

If a number is a whole what part of the fraction must it make?

10 is the whole, 12 the part. Write the fraction. ($\frac{12}{10}$.)

2 is the part, 6 the whole. Write the fraction.

9 is the whole, 8 the part. Make a "ratio to" expression.

12 is the part, 9 the whole. Ask a "part of" question.

$\frac{8}{3}$. Which of the terms is the part?

$\frac{3}{4}$. Which of the terms is the whole?

Many similar exercises along each of these lines should be given in rapid work.

C. Diagrams in "Whole and Part" Problems

1. Aim

The aim of this work is to give the pupil a means of expressing graphically the conditions and relations in "whole and part" problems of all kinds. Oral and written tests may fail to reveal imperfect percentage imagery, but outline by diagram shows without question that the child's conception and interpretation of any given problem and of the relations of its factors are or are not accurate. There is no subject in the arithmetic in which power to analyze is so essential as in percentage, hence the value of this training in graphical analysis.

2. Form of the Diagram

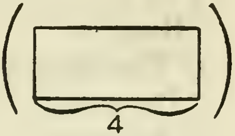
This must be *taught*. There is no way in which an arbitrary scheme for diagramming may be developed.

"3 is what part of 4?" ("$\frac{3}{4}$.")

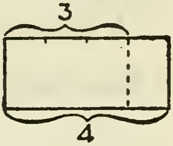
"Which is the part?" "Which is the whole?"

"We will now make a picture of this problem."

"First we will make a picture of the whole."



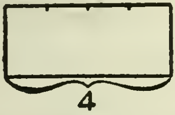
"We will now show the part."



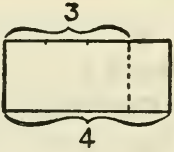
NOTE 1.—There is but one picture.

The first diagram above shows the first stage, the second diagram the completed stage of this picture.

The *whole*—



The whole and the part—



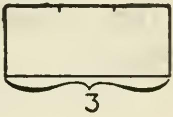
This second diagram shows the first one completed. Where the diagram is made in practice on paper or on the blackboard there is but the one picture—the second as shown above.

"1 is what part of 3?"

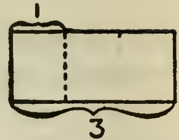
"Which is the whole? The part? The value?"

"You may make the picture."

The first stage of the picture—

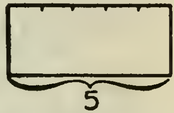


The completed picture—

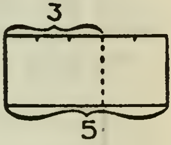


3 : 5
"Read it."
("The ratio of 3 to 5.")
"The whole?" "The part?" "The value?"
("$\frac{3}{5}$")
"You may make the picture."

The first stage of the picture—

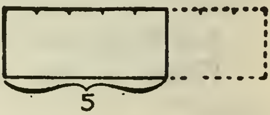


The completed picture—

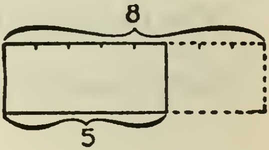


8 : 5
"Read it."
("The ratio of 8 to 5.")
"The part?" "The whole?" "The measure?"
("$\frac{8}{5}$")
"You may make the picture."

The first stage—



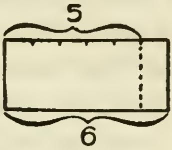
The completed diagram—



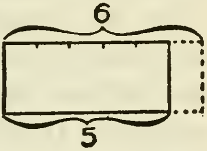
NOTE 2.—The teacher will observe from the above illustrations of diagrams that the *whole* is always represented by a rectangle with lines *entire.*

The *part* is set off by a broken line or by a series of broken lines.

The ratio of 5 to 6 is shown in—



The ratio of 6 to 5 is shown in—



The broken line with its brace shows the part.

Several additional exercises in "part of" and in "ratio to" should be given.

Exercises should be given also from oral dictations which give simply the part and the whole as follows:

6 is the part, 8 the whole. Diagram it.

9 is the whole, 12 the part. Write a ratio expression and make the diagram.

3 is the part, 2 the whole. Write a "part of" question and diagram it.

The diagram will be a regular part of the percentage work hereafter although there should be the understanding that the child will make a diagram of a problem only when the teacher specifically requires one.

3. Interpretation of "Part Of" Diagrams

This is "part of" diagram work in reverse form. The teacher makes diagrams (complete forms only) and asks pupils to find from this diagram what the "part of" question was that she had in mind when she made the picture. The child writes the "part of" question which the diagram illustrates. This tests the pupils "part of" imagery. If it means nothing to him after his experiences in graphical expression in the subject, he has no "part of" imagery. If his interpretation is doubtful or hesitating, his imagery is indistinct. If he interprets it at once and correctly, the "part of" training has borne fruit.

4. Interpretation of "Ratio To" Diagrams

This is diagram work in reverse form. The teacher makes diagrams (completed form only) on the blackboard for pupils to interpret into the "ratio to" expressions which were in mind in drawing the diagrams. This is a test of the pupil's "ratio to" imagery.

D. Part and Whole in Division

All division problems involve relations of part and whole. When we write $2 \div 3 = \frac{2}{3}$, the quotient means two toward a group of 3. 3 is the whole (the full group), 2 is the part.

1. Aim.—The aim in this division step is to develop that the dividend is always a *part;* the divisor a *whole;* the divisor, the whole, is always the denominator; the dividend, the part, is always the numerator.

2. The Development

3 ÷ 4

"What is the whole?" "What is the part?" "The quotient?"

The pupil readily infers and replies that 4 is the whole and 3 the part. The dictation is written on the blackboard, the questions are asked by the teacher, pupils reply orally, and then the quotient is written on the blackboard by some pupil.

Additional Exercises.

5 ÷ 6 =, 8 ÷ 9 =, 9 ÷ 5 =, 4 ÷ 7 =, 9 ÷ 4 =, 7 ÷ 8 =, 6 ÷ 10 =, 10 ÷ 6 =, etc.

The following questions with each exercise will hold the attention upon the facts which the step aims to develop:

Which is the part? The whole?

What is the dividend? The divisor?

Which do we use for the numerator? (The dividend.)

What do we use for the denominator? (The divisor.)

Where in the quotient fraction do we put the dividend? Where the divisor?

As to part and whole what is the divisor? What the dividend?

These questions and any others that may be used for the same purpose are designed to bring out strongly—

That the dividend is a "part" and a numerator.

That the divisor is a "whole" and constitutes the only possible denominator.

That the denominator is a whole and the divisor.

That the numerator is a part and the dividend.

3. Diagrams in Division

The factors of division, therefore, having the relations of part and whole, the diagram scheme for "part of" and "ratio to" applies unchanged.

Note.—As to the single rectangular unit to represent groups (6, 4, 2, 8, etc.), the age of the pupil must be taken into account. In the primary grades it was necessary that the objects used to show group units (6, 5, 3, etc.) be single things, so that the construction would show as a unit and at the same time as made up of the ones by which its numerical value was measured. A 4 was constructed of four, equal, distinct units.

In percentage, however, the age and training of the pupil alters conditions. He is old enough to have acquired the number concept so that he is able intelligently to "consider" a whole (a single rectangular unit) as representing a 4, a 6, a 12, or any other group unit.

3 ÷ 4 =

"Which is the whole?" "The part?" "The quotient?"

"What is the numerator?"

("3.")

"The denominator?"

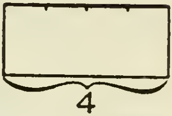
"What is the dividend?"

("3.")

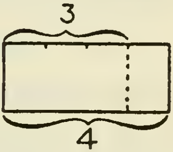
"The divisor?"

"You may make a picture."

The first stage—



The completed picture—

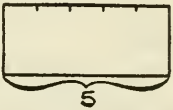


7 ÷ 5 =

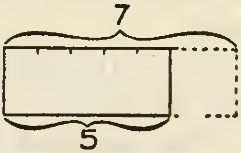
The questions are asked as above.

You may diagram.

The first stage of the diagram—



The completed diagram—



Several exercises should be given, some with proper fraction quotients and some with quotients that are improper fractions.

4. *Interpretation of Division Diagrams*

The teacher puts a series of division diagrams (the completed form only) on the blackboard. Pupils are required to write the division expression that each illustrates.

5. *Part Of, Ratio To, and Division—Mixed Exercises*

These are exercises in which the teacher puts fractions or diagrams on the blackboard for pupils to write from each, at one time a "part of" question, again a "ratio to" expression, and again a division expression.

a. INTERPRETING FRACTIONS

The teacher writes a series of fractions ($\frac{2}{3}$, $\frac{5}{6}$, $\frac{7}{4}$, $\frac{10}{9}$, $\frac{15}{16}$, $\frac{16}{15}$, $\frac{12}{4}$, $\frac{4}{12}$,) etc. on the blackboard and asks—

(*a*) for a written "part of" question for each;

(*b*) (after the "part of" questions have been written), for a "ratio to" expression for each; and

(*c*), in the same way, for a division expression for each.

The questions which have been suggested must be used, especially the two, Which is the part? and Which is the whole? These two questions should be applied to each fraction in the exercise and to other fractions written for the sake of the two questions. The value of the questions is in helping to enforce the very important fact that the denominator is *always* a *whole.* This is fundamental in percentage.

b. Interpreting Diagrams

The teacher makes a series of diagrams (completed form) on the blackboard and asks the pupils—

(*a*) for the "part of" questions which they illustrate;

(*b*) for the division expressions which they represent; and

(*c*) for the corresponding "ratio to" expressions.

c. Making Fractions and Percentage Expressions

5, 3.

"I have these two numbers. 3 is the part, 5 the whole."

"Write a division fraction." "Write the division expression."

("$\frac{3}{5}$, 3 ÷ 5")

"The two numbers are the terms of a 'ratio to' expression."

"5 is the part, 3 the whole." "Write the fraction and the expression."

("$\frac{5}{3}$, 5 : 3.")

7, 3.

"7 is the whole, 3 the part." "Write a 'part of' fraction and then the expression."

("$\frac{3}{7}$, 3 is what part of 7?")

"We will say that these numbers are the terms of a division expression." "Write the fraction and then the expression."

("$\frac{3}{7}$, $3 \div 7 =$.")

Several exercises of this kind should be given—

4, 8; 7, 3; 9, 4; 2, 5; 10, 8; 12, 9; 7, 5; etc.

The special aim in such exercises is the *fraction* part. It tests the pupil's practical grasp of the very important fact that a *denominator* must be a *whole.*

E. Part and Whole in "Than" Expressions

1. The Plan

The part and whole nature of problems in "than"—problems involving conditions of greater than, less than, smaller than, larger than, etc.—is developed on much the same plan as that used under "part of" and "ratio to."

The questions suggested in those steps are applicable and must be used here. The points to which attention must be given are—

The recognition of the whole and the part or parts.

The writing of the fraction values.

The construction of diagrams.

2. *The Development*

"8 is how much less than 9?"

"What is the whole?" ("9.")

Pupils generally agree on 9 as the whole. The *part*, however, is not recognized until the teacher calls attention to the form of the problem—"How much *less*?"

"What is the part?"

The 8 is 1 less. The *part* is 1.

"What is the fraction value of the part?" ("$\frac{1}{9}$.")

"Why is 9 the denominator?" ("Because it is the whole.")

This means that 8 is $\frac{1}{9}$ less than 9, which is true.

"9 is how much greater than 8?"

"What is the whole?" ("8.")

"What is the part?" ("1.")

"What is the measure of the part?" ("$\frac{1}{8}$.")

"Why do you make 8 the denominator?" ("Because it is the whole.")

Other similar exercises are given varying the question so that they involve many forms of than—smaller than, more than, heavier than, taller than, etc. The child gradually recognizes the whole from its relation to or association with *than*.

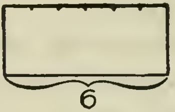
3. *The Diagrams of "Than" Expressions*

"8 is how much more than 6?"

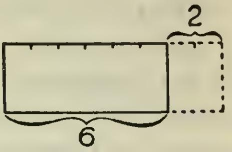
"What is the whole?" "The part?" "The measure?" "Why 6 for the denominator?"

"You may make the picture."

The first stage—



The full diagram—

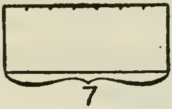


"4 is how much less than 7?"

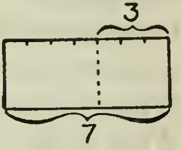
The questions as to whole, part, measure, and denominator should be used.

"You may make the diagram."

The first stage—



The completed diagram—



Several exercises must be given in such diagrams until pupils become accustomed to the *part*—that which represents the difference which the comparison in the greater than or the less than or the smaller than, etc., calls for.

F. Making Fractions from Abstract Related Numbers

1. Aim.—The aim in this step is to concentrate attention again and as strongly as possible upon the one fundamental principle of percentage—that the denominator of a fraction must be a whole and that no other number than the whole can be used as a denominator.

This principle is a rule that must at the proper time be formulated, memorized, and recited.

2. *The Development Plan*

The teacher writes a group of numbers on the blackboard marking one of the numbers—2, 4, $\underset{\text{x}}{3}$, 5. She then calls the attention to the numbers and states that one of them, the one that is marked, is a *whole* and that it is the *only* whole in the group. She then asks the pupils to use the numbers in making fractions and to make as many fractions as possible from them. This is one of the practical tests of the pupil's grasp of the principle that has been repeatedly urged, that no number except one which represents a *whole* may be used for a denominator. In the group given above, 3 is the only one of the numbers which, under the rule, may be a denominator. Most pupils will fail to rise to the occasion fully on the first test. The plan is to write the numbers, mark one of them as a whole, state that the marked number is a whole and the only whole in the group, and leave the pupils to make the fractions without further suggestion. It impresses a pupil very much more strongly to be "caught" in a violation of the rule about the *whole* being the only number that may be used for a denominator than any amount of teaching or suggesting. When they have written their fractions, one or more of the pupils should copy their work on the blackboard. It is then that the teacher calls attention to the observances and violations of the rule. In the group given above the possible fractions are $\frac{2}{3}$, $\frac{4}{3}$, $\frac{3}{3}$, and $\frac{5}{3}$. Another exercise is given—$\underset{x}{5}$, 8, 4, 7. Pupils are again asked to make all the possible fractions. These are $\frac{5}{5}$, $\frac{8}{5}$, $\frac{4}{5}$, and $\frac{7}{5}$.

The question should be asked in every such exercise, Why do you use —— for the denominator in every fraction? (It is the whole.)

Three or four exercises should be given before taking up the matter of what the fractions mean.

3. *What the Fractions Mean*

After three or four exercises the teacher takes up, in connection with the work of fraction making, the meaning of each fraction.

9, 7, $\underset{x}{5}$, 3, 1.

"Make all possible fractions from these numbers, 5 being a whole."

("$\frac{9}{5}$, $\frac{7}{5}$, $\frac{5}{5}$, $\frac{3}{5}$, $\frac{1}{5}$.")

"What is $\frac{5}{5}$ the fraction of?"

("It is the fraction of the *whole.*")

"What is $\frac{9}{5}$ the fraction of?"

("Of a part.")

"What is $\frac{7}{5}$ the fraction of?"

("Of a part.")

(The teacher will observe that in the fractions from a group of this kind there is one that is the fraction of the whole, the others are fractions of parts.)

4. *The Rule as to Fraction Making*

The "rule" is alluded to above is now formulated and made a matter to be recited upon in the future very frequently—

The denominator of a fraction *must be* a whole and no other number than a whole may be used for a denominator.

5. *Additional Exercises*

Many exercises in fraction making as above must be given. These exercises must include statements of what each fraction means and also occasional statements of the "rule." "*The fraction of the whole*" *especially* must be noted.

$3, \underset{x}{4}, 6, 5;\ 4, 6, \underset{x}{2};\ 8, 2, \underset{x}{9}, 4;\ 5, 8, 3, \underset{x}{6}, 4;\ \underset{x}{5}, 3;$
$\underset{x}{3}, 9, 8, 7;\ 5, \underset{x}{6}, 9;\ \underset{x}{2}, 3;\ \underset{x}{9}, 2, 4;\ 9, 2, \underset{x}{4};\ 9, \underset{x}{2}, 4;$ etc.

NOTE.—This completes the development work and training preliminary to the regular work of percentage applications. The time required for the work to this point should not be more than 10 lesson periods of 20 minutes each with occasional 10-minute reviews later after applied percentage work is taken up.

SUMMARY.

The pupil learns to see in "percentage" a subject in which the measurements are entirely of "part and whole" relations—the relations of parts to a whole. These measurements are fractional expressions in which the denominator in each case must be the whole. (*A*.)

The natural "part and whole" fractions are in the *common* fraction form. The *decimal* (per cent) form is artificial, it being necessary for the individual to interpret it in each case by mental translation to a common fraction. The introductory work in "percentage" must, therefore, be through the use of the common fraction, the decimal form being a later special development. (*A*, Note.)

The "part and whole" relations are those which find expression in "part of," "ratio to," "than" (greater than, less than, etc.), and division. The pupil learns to note these relations and to express their measurements in common fraction form. He is given experiences in changing common fractions now to one and now to another of the formal "whole and part" expressions—part of, ratio to, etc. (*B*.)

All expressions of the relations of parts to wholes may be shown in diagram form. The pupil learns to make such diagrams. (*C*.)

CHAPTER II

APPLIED WORK IN PERCENTAGE

Topics:—Part and Whole Applied to—Loss and Gain; Trade Discount; Collecting; Fire Insurance

A. Aims

The plan to be followed in this development of applied work is to regard each class of problems as a unit and to take them up one after the other as separate steps. These classes of problems are to be taken in the following order—loss and gain, trade discount, collecting, selling commission, buying commission, fire insurance, tariff duties, and property taxes.

Later, what may be regarded as percentage *concepts* will be treated—that form of solution in which, for computation purposes, all terms are translated into corresponding *general* terms—base, rate and percentage, amount, and difference.

B. Loss and Gain

1. The Factors in Loss and Gain

"I buy for $6 and sell for $7." (Written on the blackboard.)

"In this statement, what facts are known?"

The pupils will find these facts—the cost, $6; the selling price, $7; and the gain or profit $1.

"How many facts?"

("Three,—the cost, selling price, and gain.")

"I sell for $5 and lose $3."

"In this problem, what facts are known?"

("The selling price, $5; the loss, $3; and the cost, $8.00.")

"How many facts?"

("Three,—selling price, loss, cost.")

Several simple loss and gain problems are given as above and the pupils are asked to point out the known facts and state how many such facts there are.

This will develop that in a loss problem there are three and *only* three facts—cost, loss, and selling price; and that in a gain problem there are three and *only* three facts—cost, gain, and selling price.

These developments are summarized into statements—

"What are the items in a loss problem?"

("The items in a loss problem are cost, loss, selling price.")

"What are the items in a gain problem?"

("The items in a gain problem are cost, gain, selling price.")

"How *many* items in a loss or a gain problem?"

("A gain or a loss problem has three items.")

2. *The Fractions in Loss and Gain*

a. The Written Fraction

The pupils are now told that in every loss and in every gain problem there are *parts* and a *whole*, that the cost is *always*, *always* a *whole*, and that the other two items in a problem of either of these kinds are *parts*. These are arbitrary facts and not possible of development, hence the pupils are *told* directly.

"Gained $3 by selling for $8."

"What kind of a problem is this?"

("It is a gain problem.")

"What are the items?"

("Gain, $3; selling price, $8; cost $5.")

"How many items?"

"What is the whole?"

("$5.")

"Why?"

("Because it is the cost.")

"You may make fractions from the item numbers."

(The pupils should make the fractions without further suggestions. After they have written the fractions, several of the pupils should copy on the blackboard what they have written. The teacher asks of each of these sets of fractions whether the right number has been used for the denominator. This brings the child to the application of his "rule" (see Chapter I, *F 4*) about *whole* being the only possible denominator.)

(The fractions in this case should be $\frac{3}{5}$, $\frac{5}{5}$, $\frac{8}{5}$.)

"What is $\frac{3}{5}$ the fraction of?"

("Of the gain.")

"What is $\frac{5}{5}$ the fraction of?"

("Of the cost.")

"What is $\frac{8}{5}$ the fraction of?"

("It is the fraction of the selling price.")

"The cost of a box was $4. I sold at a loss of $1."

"What kind of problem?"

("A loss problem.")

"The items?"

("Cost, $4; loss $1; selling price $3.")

"How many items?"

"What is the whole?" "Why?"

"You may make the fractions."

("$\frac{4}{4}$, $\frac{1}{4}$, $\frac{3}{4}$.")

"Why 4 for the denominator?"

"What is $\frac{4}{4}$ the fraction of?" "$\frac{1}{4}$?" "$\frac{3}{4}$?"

"State the *rule* about the denominator in a 'part and whole' problem."

b. ORAL WORK IN FRACTIONS

After a few exercises as above in written fractions, oral drills should be given in *mental* fraction work—this work to be as rapid as possible. The problems should be given in brief form and may be in abstract numbers instead of in money language.

"Bought for 9 and sold for 10." "Kind of problem?" ("Gain.") "Give fraction of selling price." "Fraction of gain." "Fraction of the whole." "The rule about the denominator?"

"Bought for 10 and sold for 9." Kind of problem? "Fraction of selling price." "Fraction of the whole." "Fraction of the loss?" "Rule?"

In the same way—

"Sold for 8 and lost 2."
"Sold for 9 and gained 3."
"Bought for 6 and sold for 4."
"Bought for 4 and sold for 6."
"Sold for 20 and gained 4."
"Bought for 20 and lost 4."

The aim of such work is, of course, rapid selection of the proper number for the denominator.

3. *Statements in Loss and Gain*

The pupils are now taught to make a full written statement in form as follows, using the transaction, Sold for \$3 losing \$1, the statement to show all the facts—

$(\frac{4}{4})$ 1 = Cost = \$4
$\frac{1}{4}$ = Loss = \$1
$\frac{3}{4}$ = Selling Price = \$3

The transaction furnishes, practically, the three money items on the right of the statement and the 1 (the whole) on the left side. The child makes the fractions $\frac{4}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$ from the right hand items.

"Gained 4¢ by selling for 20¢."

The questions suggested in the exercises above (in *2*) relating to the kind of problem, the items, number of items, whole, etc., are asked and the fractions called for. The *rule* as to denominator being a whole is called for if thought best.

"You may now make the full statement."

$$\begin{aligned} (\tfrac{16}{16}),\ 1 &= \text{Cost} &&= 16¢ \\ \tfrac{4}{16} &= \text{Gain} &&= 4¢ \\ \tfrac{20}{16} &= \text{Selling Price} &&= 20¢ \end{aligned}$$

These loss and gain statements should be written in the order as above—cost, first; gain or loss, second; selling price, third. The *fractions* should always be placed in the same part of the statements—on the left side, as above, is preferable.

In the exercise above, the fractions are $\frac{16}{16}$, $\frac{4}{16}$, etc. Pupils hereafter should be required to put fractions into the lowest terms unless for development or other reasons a higher form should be desired by the teacher in special exercises. The above fractions should be written ordinarily as $\frac{4}{4}$, $\frac{1}{4}$, $\frac{5}{4}$. The fraction of the whole in the parenthesis will eventually be discontinued. The proper form to be used is 1 (the whole) because it meets our need in using decimals later.

Instead of using the full words cost, gain, loss, selling price, it is desirable on account of the brevity to use abbreviations—C., G., L., and S. P.

"The selling price was 75¢, the loss 15¢."

The questions are asked and then the statement called for. The pupil first enters the items from the problem—

$$\begin{array}{rll} 1 & = \text{C.} & = 90¢ \\ & = \text{L.} & = 15¢ \\ & = \text{S. P.} & = 75¢ \end{array}$$

He now makes the fractions ($\frac{90}{90}$, $\frac{15}{90}$, $\frac{75}{90}$), reduces them to their lowest terms ($\frac{6}{6}$, $\frac{1}{6}$, $\frac{5}{6}$), and completes the statement by entering the fractions. This reduction of the fractions will doubtless be mental—the child *writing* only the reduced forms.

"Sold for $16 and made a profit of $\frac{1}{3}$."

The questions are asked—

"What kind of a problem?" The whole?

"The items?"

("Only one of the three is given.")

"You may make the statement."

$$\text{First stage}\left\{\begin{array}{rll} 1 & = \text{C.} & = \\ & = \text{G.} & = \\ & = \text{S. P.} & = \end{array}\right\}$$

This outline part of a statement should always be made first.

$$\text{Second stage}\left\{\begin{array}{rll} (\frac{3}{3}),\ 1 & = \text{C.} & = \\ \frac{1}{3} & = \text{G.} & = \\ \frac{4}{3} & = \text{S. P.} & = \$16 \end{array}\right\}$$

The outline is now filled in from the problem as shown above. The outline now shows what the problem practically tells us.

$$\begin{array}{l}\text{The statement}\\ \text{as it appears}\\ \text{at completion.}\end{array}\left\{\begin{array}{rll} (\frac{3}{3}),\ 1 & = \text{C.} & = \$12 \\ \frac{1}{3} & = \text{G.} & = \$4 \\ \frac{4}{3} & = \text{S. P.} & = \$16 \end{array}\right\}$$

The statement is now completed as shown here. The pupil *writes out* the analysis of the computations as follows:

$$\begin{array}{l} \frac{4}{3} = \$16 \\ \frac{1}{3} = \frac{1}{4} \text{ of } \$16 = \$4 \\ \frac{3}{3} = 3 \times \$4 = \$12. \end{array}$$

NOTE.—Practically the pupil will complete his statement at once, computing mentally. There could be no objection to this. The analysis, however, must be written out. This analysis, at this time, is for the sake of the analysis. Problems will present themselves later in which analysis will be needed.

Three or four exercises like the above in which the pupils make completed solutions should be given before taking up the diagram feature of loss and gain work as outlined below. In each of these, the pupil should be required to proceed systematically—

(1) By making the statement form with its equality signs and the 1 (whole) as shown in "first stage" above.

(2) By putting into the statement all that the problem tells, as in "second stage."

(3) By making the computations in analysis form as shown above and then completing the statement so that it will be in "at completion" form as above.

The pupil's statement and computation work will appear as follows when he completes a solution:

"I lost 5¢ by selling at $\frac{1}{8}$ below cost."

$\left(\frac{8}{8}\right)$, 1 = C. = 40¢ *
$\frac{1}{8}$ = L. = 5¢
$\frac{7}{8}$ = S. P. = 35¢

$\frac{1}{8} = 5$¢
$\frac{7}{8} = 7 \times 5$¢ $= 35$¢
$\frac{8}{8} = 8 \times 5$¢ $= 40$¢

---

* The statement form as given here is used in all the applications of percentage—Loss and Gain, Trade Discount, Collecting, Selling Commission, etc. The teacher should *insist* upon the use of this form of solution arrangement in *every problem*. It is a short, simple, compact, complete form, providing for the display of every fact relating to the problem. If the problem asks for the loss (in a loss problem), the rate of loss, the cost, or any other fact, the statement when completed shows it. It answers any question. Again, such

a statement in full is a check upon errors in computation or reasoning. If there is an error, the completed statement will be out of balance. Again, it is as brief as is compatible with accuracy. Again, under it the pupil's work is arranged in orderly manner.

4. Loss and Gain Diagrams

As loss and gain problems are in the "whole and part" class, the diagram work under that subject was sufficient preparation for the use of diagrams here. The diagrams simply show the *whole* and the two *parts* (the selling price and gain, or the selling price and loss).

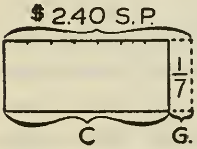
"The selling price was \$2.40, the gain $\frac{1}{7}$."

The pupil, if at the blackboard, is required to "talk" as he works at his diagram—

"This is a gain problem. I first make a picture of the whole which is the cost."

After he has drawn the "whole" he continues—

"I must now make the selling price and the gain. This is a gain problem, therefore for the selling price I must *add on* in the picture."



$(\frac{7}{7})$, $1 = \text{C.} \quad = \$2.10$

$\frac{1}{7} = \text{G.} \quad = \$.30$

$\frac{8}{7} = \text{S. P.} = \$2.40$

$\frac{8}{7} = \$2.40$

$\frac{1}{7} = \frac{1}{8}$ of $\$2.40 = \$.30$, Gain.

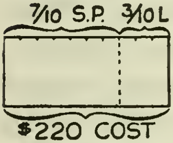
$\frac{7}{7} = 7 \times \$.30 = \$2.10$, Cost.

(The oral work by the pupil as he is making his diagram should be a feature of the construction. It should not be overlooked. The same oral statements of what is being done should be a part of the work of writing the *analytical solutions* which follow the making of the "2d stage" of the written outline of the problem.)

"The loss was $\frac{3}{10}$, the cost $220." "Diagram."

The pupil who is sent to the blackboard "talks" as suggested above—

"This is a loss problem. I must first make a picture of the whole—the cost. I must now make the selling price and the loss. This is a loss problem, therefore, for the selling price, I must *take off* in the picture."



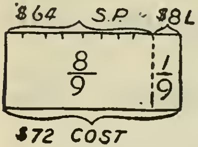
$$\left(\tfrac{10}{10}\right),\ 1 = \text{C.} = \$220$$
$$\tfrac{3}{10} = \text{L.} = \$66$$
$$\tfrac{7}{10} = \text{S. P.} = \$154$$

$$\tfrac{10}{10} = \$220$$
$$\tfrac{1}{10} = \tfrac{1}{10} \text{ of } \$220 = \$22$$
$$\tfrac{3}{10} = 3 \times \$22 = \$66, \text{ L.}$$
$$\tfrac{7}{10} = 7 \times \$22 = \$154, \text{S.P.}$$

"Lost $8 on the sale of a cow for $64." "Diagram."

The pupil at the blackboard gives the oral work as suggested—

"This is a loss problem. I first make a diagram of the whole—the cost. I now show the selling price and the loss. This is a loss problem, therefore for the selling price I must *take off* in the diagram."



$(\frac{9}{9})$, 1 = C. = $72

$\frac{1}{9}$ = L. = $8

$\frac{8}{9}$ = S. P. = $64

$\frac{8}{72} = \frac{1}{9}$ = Fraction of L.

$\frac{64}{72} = \frac{8}{9}$ = Fraction of S. P.

5. *Proportions in Diagrams*

In problems where fractions are given to find the money gain or loss, or other money items, the diagram need not be drawn to exact scale so that the fractional parts shall show to scale. The diagrams should be freehand and when the fractional parts can be easily shown in something like proper relative size they should be drawn with a degree of care. When, however, a fractional part is so small that any attempt at a scale drawing would require effort out of proportion to the value of the result, any reasonable diagram should be accepted provided the pupil writes on it the fraction that each part of the diagram is intended to represent. The diagram above (Lost $8 by selling for $64) shows what each part represents. The parts, however, are out of proportion. The diagram was purposely so drawn to illustrate how to indicate relative sizes of parts in percentage when a drawing is not to scale. The lack of proper proportion in this diagram does no manner of harm because the fractional values are plainly stated. Diagrams in percentage are not for training in drafting or for aid in "getting an answer," but to show to the teacher what the child is thinking about in the problem. The diagram under discussion tells us all that could be told by the most perfectly executed scale drawing.

The teacher, however, should require that the diagrams be executed with neatness—the lines straight and the corners reasonably perfect.

a. Additional Exercises

Many additional exercises must be prepared but the teacher must pass to the next step—Trade Discount—as soon as the pupils show reasonable accuracy in the diagram and the analysis work with simple problems such as have been given in the illustrative work above.

The pupil will not show discrimination and power in either loss or gain or trade discount work until the point is reached where he must deal with mixed exercises—exercises in which the problems are sometimes in loss and gain and sometimes in trade discount. There is less necessity for concentration of attention where the individual is dealing with problems in which he is not compelled to classify each problem before he attempts solution. The mixed exercises compel the classification of a problem before attempting a solution.

b. REVIEWS

The teacher must not overlook the fact that in all percentage work, as well as in all other subjects, reviews of all previous work in the subject must be regularly given. With this work in loss and gain and with the work in the subjects which follow, there should be occasional reviews of the introductory percentage work.

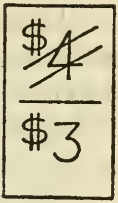
C. Trade Discount

Problems of this class relate to matters which touch every child's daily life. "Fire and water sales," "pre-inventory sales," "anniversary sales," etc., with their "marked down" tags are familiar to everyone. The work follows the plan outlined for Loss and Gain.

1. The Factors in Trade Discount

a. THE INTRODUCTION

The introduction to this should be through representations of "marked down" tags rather than through formally stated problems.



This is placed on the blackboard by the teacher.

"Do you remember ever having seen anything that looks like this? One or more pupils may recognize that it represents a tag on a piece of goods in the show window of a store. If not, the suggestion should be made "I saw something of the kind on a hat down town in a show window." This will open up the subject fully.

"What does it mean?"

From this question it is brought out that the article was marked to sell for $4 but not being sold the merchant put the sale price at $3 hoping to find a customer at that price. Several other tag representations are made on the blackboard and discussed in the same way—as to what it signifies. The questions as to the "items" (see *B 2 a*) should be asked about each tag.

"From this tag, *how many* items do we know?"

("We know three items—the marked price, $4; the proposed selling price, $3; and the amount taken off for the sake of a sale $1.")

b. Discount and Trade Discount—The Terms

These terms must be defined and used.

"People call this amount taken off, the *discount.*"

"Discount means *to take off*. We use the word also as the name of the amount that is taken off—*the* discount."

Pupils should now be questioned about each of the tags—

"What is the discount here?" "What is the discount here?" etc.

Pupils should be given also the information that there is more than one kind of discount and therefore people speak of this kind, where something is taken off the price of goods when sold or offered for sale, trade discount. The trade discount on the hat marked down from $4 to $3 is $1. Pupils will be permitted to speak of this ordinarily as *discount*, but the question, "What kind of discount?" must be very frequently asked so that the full term, trade discount, will be established in the vocabularies.

This term discount (and with it, as suggested, the term trade discount) must now be put into constant use in the class. The tag representations on the blackboard are examined and the names marked price, trade discount, and selling price are applied to the items. The questions are—

What is the marked price here?

What is the discount here? etc.

What is this (pointing to a marked price)?

What is this (pointing to a selling price)? etc.

NOTE 1.—The pupil should learn to state in answer to the question, How many trade discount items are there? the number and the names of the trade discount items—There are three trade discount items, marked price, trade discount, and selling price.

2. *The Fractions in Trade Discount*

a. THE WRITTEN FRACTION

The pupil is now told that in a trade discount tag or problem there are *parts* and a *whole* as in loss and gain, that the *marked price* is *always*, *always* the *whole* and that the other two items are *parts*.

"Goods were marked down from $6 to $4."

"What kind of a problem?" (A trade discount problem.")

"What items and how many?" (Three—marked price, $6; discount, $2; selling price, $4.")

"What is the whole?" ("The marked price, $6.")

"You may make the fractions."

The fractions are written and several pupils write their fractions on the blackboard.

These are discussed as to whether or not the denominators are right and why.

The fractions in this case are—$\frac{6}{6}$, $\frac{2}{6}$, $\frac{4}{6}$.

The questions are asked as to what each is the fraction of. These questions bring out that $\frac{6}{6}$ is the fraction of the marked price;

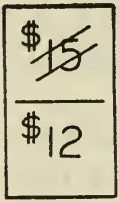
$\frac{2}{6}$, the fraction of the discount; $\frac{4}{6}$, the fraction of the selling price.

The above development and questions indicate the form of the work. Several exercises should be given, some from problems stated regularly and some given in the form of tag representations.

b. Oral Work in Fractions

Exercises should also be given in oral statements of the fractions in problems.

The oral statements from this are—



"This is a trade discount problem.
The marked price is the whole.
$\frac{15}{15}$ is the fraction of the marked price.
$\frac{12}{15}$ is the fraction of the selling price.
$\frac{3}{15}$ is the fraction of the discount."

Several such exercises should be given.

3. Statements in Trade Discount

These are in the very same form as those for loss and gain. The statements must be made in the order to correspond with the order suggested for the statements in loss and gain—1st, the items in order with their equality signs on both sides; 2d, entering on this form the facts from the problem; 3d, the computations, if any, in analysis form; and 4th, the completion of the form from the computations. (See *B, 3*.)

"Sold goods for $5 after discounting $2."

There must be, first, the questions to bring out—the kind of problem (trade discount), the names and number of the items, and the whole. (See *a* above.)

"You may make the statement."

$$\left.\begin{array}{rll} 1 &= \text{M. P.} &= \\ &= \text{D.} &= \\ &= \text{S. P.} &= \end{array}\right\}$$

This is the first stage of the statement.

$$\left.\begin{array}{rll} 1 &= \text{M. P.} &= \$7 \\ &= \text{D.} &= \$2 \\ &= \text{S. P.} &= \$5 \end{array}\right\}$$

This shows how the statement looks after entering the facts from the problem.

$$\left.\begin{array}{rll} (\tfrac{7}{7}),\ 1 &= \text{M. P.} &= \$7 \\ \tfrac{2}{7} &= \text{D.} &= \$2 \\ \tfrac{5}{7} &= \text{S. P.} &= \$5 \end{array}\right\}$$

This shows how the statement looks at completion.

There would be no analysis computation for a problem of this kind because the only work necessary is the formation of the fractions and their reduction to lowest terms when such reduction is necessary. The statements (oral) should occasionally be required relating to what each of the fractions is the fraction of:

$\frac{2}{7}$ is the fraction of the discount (in the statement above).

$\frac{7}{7}$ is the fraction of the marked price, etc.

"Sold for $15.75 after giving a discount of $\frac{1}{8}$."

The questions (if it is a blackboard exercise) bringing out the kind of problem, the names and number of the items, and the whole, should be asked.

"You may make the statement." *

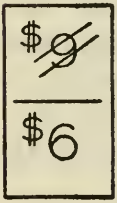
$$\begin{array}{lll} (\tfrac{8}{8}),\ 1 & = \text{M. P.} & = \$18. \\ \quad\ \tfrac{1}{8} & = \text{D.} & = \$\ 2.25. \\ \quad\ \tfrac{7}{8} & = \text{S. P.} & = \$15.75 \\ \hline \quad\ \tfrac{7}{8} & = \$15.75 & \\ \quad\ \tfrac{1}{8} & = \tfrac{1}{7} \text{ of } 15.75 = \$2.25, \text{ D.} & \\ \quad\ \tfrac{8}{8} & = 8 \times 2.25 = \$18, \text{ M. P.} & \end{array}$$

4. *Trade Discount Diagrams*

As the diagram plan in all "whole and part" work is the same it is unnecessary to do more than to show how the plan applies in trade discount.

(This is a problem stated in "tag" form.)

"Diagram and solve."



The pupil in solving at the blackboard before the class "talks" as he proceeds—

"This is a trade discount problem. The whole is $9, the marked price."

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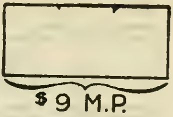
* The statement form as given here and in loss and gain is the form that should be insisted upon in every written solution. (See footnote, p. 775.)

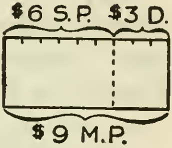
"I will make the diagram."

"I first make a picture of the whole."



(The first stage of the diagram.)

"I show the discount. It is a discount problem therefore I must take off."



The diagram when completed.

$(\frac{3}{3})$, 1 = M. P. = \$9
$\frac{1}{3}$ = D. = \$3 $\quad \frac{3}{9} = \frac{1}{3}$ = fraction of the D.
$\frac{2}{3}$ = S. P. = \$6 $\quad \frac{6}{9} = \frac{2}{3}$ = fraction of the S. P.

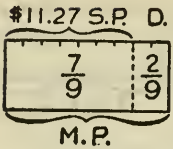
"Sold at a discount of $\frac{2}{9}$. The selling price was \$11.27."

"Diagram."

("This is a trade discount problem. The whole is the marked price." "I will make a picture of the whole." "I will show the discount. It is a discount problem therefore I take off."

$$\left.\begin{aligned}(\tfrac{9}{9}),\ 1 &= \text{M. P.} = \\ \tfrac{2}{9} &= \text{D.} \quad\ \ = \\ \tfrac{7}{9} &= \text{S. P.} = \$11.27\end{aligned}\right\}\ \text{With the facts from the problem.}$$

$$\left.\begin{aligned}(\tfrac{9}{9}),\ 1 &= \text{M. P.} = \$14.49 \\ \tfrac{2}{9} &= \text{D.} \quad\ \ = \$\ \ 3.22 \\ \tfrac{7}{9} &= \text{S. P.} = \$11.27\end{aligned}\right\}\ \text{The statement when completed.}$$



$\frac{7}{9} = \$11.27$

$\frac{1}{9} = \frac{1}{7}$ of $\$11.27 = \$1.61$

$\frac{2}{9} = 2 \times \$1.61 = \$3.22$ D.

$\frac{9}{9} = 9 \times \$1.61 = \$14.49$ M. P.*

NOTE 2.—When the teacher wishes to have a pupil make a diagram of the problem she indicates it by the word "Diagram" or by the simple initial "D." If D or Diagram is not written with the problem, the pupil gives only the statement and the analytical solution.

NOTE 3.—Teachers will find, if they make the test, that most children of the age to take up this subject of percentage will be under the impression that a sale at a discount means a loss to the seller. Goods marked to sell for 60¢ and sold for 50¢ will impress them as a losing bargain for the merchant. This is doubtless merely an impression that has remained until it seems to them a fact not to be disputed. One of the duties of the teacher in this trade discount work is thoroughly to disabuse the children's minds of this idea. The question must be asked quite often, How much did he lose? If the child's reply is an admission of a loss, the teacher must state emphatically that there was a good profit. The teacher's attitude, in the absence of good evidence that there *was* or *must* have been a loss, must invariably be that there is not the slightest doubt that the merchant made a profit. It is only by this emphatic insistence upon the opposite view—that discount sales are sources of profit—that the child can be drawn away from his seemingly natural tendency to associate discounts with losses.

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*Some pupils will write this statement—$\frac{9}{9} = \$11.27 + \$3.22 = \$14.49$ M. P. This is proper and should be fully approved and accepted.

5. *Mixed Exercises in Gain, Loss, and Trade Discount*

Exercises should be given daily in the solution of mixed problems in gain, loss, and trade discount. Such exercises should continue during the development work that follows. In most of these exercises the pupil should be required to make the diagrams as well as the computations. As has been suggested, the teacher should be insistent upon the solutions being made *in every case* in the statement form as in the development work. The teacher should be insistent also upon the pupil's beginning his solution work with the statement according to the problem, This is a gain problem, the whole is the cost; or, This is a loss problem, the whole is the cost; or, This is a trade discount problem, the whole is the marked price. The habit of thus looking into a problem to determine its nature before attempting a diagram or statement helps to insure the child against needless errors like solving a gain problem as if it related to a loss or a trade discount transaction.

6. *Double Trade-Discounts*

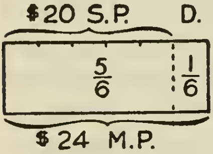
This step has to do with the problems arising from sales of goods where two discounts are given. The teacher will understand that this work with common fraction discounts is merely preparation for the regular work later with hundredths.

a. Oral Exercises

The aim here is to use very simple problems as a means of giving the child information about double discounts—what we mean by such discounts and how the second discount is computed.

"Goods marked to sell for $24 were sold at $\frac{1}{6}$ and $\frac{1}{20}$ off."

"This means that the merchant gave a trade discount of $\frac{1}{6}$. That would give a selling price of $20. He gave, however, another discount of $\frac{1}{20}$. This means $\frac{1}{20}$ of the new selling price, $20. For this latter discount, therefore, the $20 is the marked price."



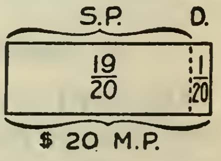
$(\frac{6}{6}),\ 1 = \text{M. P.} = \$24$
$\frac{1}{6} = \text{D.} = \$\ 4$
$\frac{5}{6} = \text{S. P.} = \$20$

$\frac{6}{6} = \$24$
$\frac{1}{6} = \frac{1}{6} \text{ of } \$24 = \$4, \text{ D.}$
$\frac{5}{6} = 5 \times \$4 = \$20, \text{ S. P.}$

The above are the diagram, outline statement, and analytical solution for first discount.

$(\frac{20}{20}),\ 1 = \text{M. P.} = \$20$
$\frac{1}{20} = \text{D.} = \$1$
$\frac{19}{20} = \text{S. P.} = \$19$

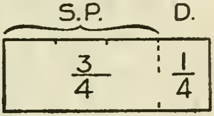
$\frac{20}{20} = \$20$
$\frac{1}{20} = \frac{1}{20} \text{ of } \$20 = \$1, \text{ D.}$
$\frac{19}{20} = 19 \times \$1 = \$19, \text{ S. P.}$



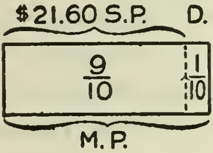
The above are the diagram, etc., for the final discount. The instruction should aim to inform the pupil how two discounts are computed—the first on the first marked price and the second on the first selling price used as a new marked price. This develops that there are *as many problems* and *as many solutions* as there are successive discounts.

"Sold goods for \$21.60 after giving successive discounts of $\frac{1}{4}$ and $\frac{1}{10}$."

("This is two trade discount problems in one." "The whole in a trade discount problem is the marked price."



"Being discount problems I 'take off' in each." "I make pictures of the two problems.")



$(\frac{4}{4})$, 1 = M. P. =
$\frac{1}{4}$ = D. =
$\frac{3}{4}$ = S. P. =

$(\frac{10}{10})$, 1 = M. P. =
$\frac{1}{10}$ = D. =
$\frac{9}{10}$ = S. P. = \$21.60

The above are the two diagrams and the first stages of the two outline statements that the pupil makes directly from the problem. The pupil is now ready to begin his solutions.

From the outline, it is readily seen that the second problem is the first to solve—

$\frac{9}{10} = \$21.60$
$\frac{1}{10} = \frac{1}{9}$ of \$21.60 = \$2.40, D.
$\frac{10}{10} = 10 \times 2.40 = \$24$, M. P.

The pupil now completes the 2d outline by putting the $24 in its place as the M. P. of that problem—$(\frac{10}{10}),\ 1 = \text{M. P.} = \$24$

$\frac{1}{10} = \text{D.} \quad = \$\ 2.40$

$\frac{10}{10} = \text{S. P.} \quad = \$21.60$

The pupil has learned that the S. P. of the first discount is always the M. P. of the second discount.

The $24, the M. P. of the second outline, is now written in the S. P. part of the first outline and the solution made.

$(\frac{4}{4}),\ 1 = \text{M. P.} = \$32$	$\frac{3}{4} = \$24$
$\frac{1}{4} = \text{D.} \quad = \$\ 8$	$\frac{1}{4} = \frac{1}{3}$ of $\$24 = \$8$, D.
$\frac{3}{4} = \text{S. P.} \quad = \$24$	$\frac{4}{4} = 4 \times \$8 = \$32$, M. P.

The above shows the first outline in its completed form and the analytical statement.

"A merchant sold goods for $24.50 after successive discounts of $\frac{1}{3}$ and $\frac{1}{8}$." "Diagram."*

("This is two problems in trade discount." "The whole in a trade discount problem is the marked price."

"In trade discount we take off.

"I will first make the diagrams.")

$(\frac{3}{3}),\ 1 = \text{M. P.} \quad = \$42.$

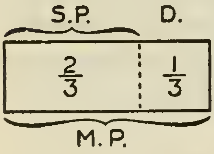
$\frac{1}{3} = \text{D.} \quad = \$14.$

$\frac{2}{3} = \text{S. P.} \quad = \$28.00$

$\frac{2}{3} = \$28.$

$\frac{1}{3} = \frac{1}{2}$ of $\$28 = \$14$, D.

$\frac{3}{3} = 3 \times \$14 = \$42$, M. P.



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* See Note 2 above.

$(\frac{8}{8})$, 1 = M. P. = $28.

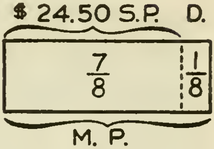
$\frac{1}{8}$ = D. = $ 3.50

$\frac{7}{8}$ = S. P. = $24.50

$\frac{7}{8}$ = $24.50

$\frac{1}{8} = \frac{1}{7}$ of 24.50 = $3.50, D.

$\frac{8}{8}$ = 8 × $3.50 = $28, M. P.



Several double trade-discount problems should be given and then single and double problems of this class should form a part of the work in mixed problems that pupils should have daily for seat practice work in gain, loss, and trade discount.

D. Collecting

1. *The Aims and the Vocabulary*

The first step in this subject is a series of explanatory illustrations of what is meant by collecting debts; what conditions make paid collectors of business debts necessary; and what we mean by collecting agent, commission, and net proceeds.

"A merchant sold goods to Mr. A for $41. Mr. A failed to pay. The merchant finally asked a lawyer to collect. The latter collected the debt and gave the merchant $38, the rest of the money being kept as his pay for the service.

The $3, the collector's pay, is called the *commission*. This commission is deducted by the collector from the amount collected and the balance is paid over to the merchant.

The $38, the amount that the merchant received from his collector, is called the net proceeds. The amount that the agent collected, $41, goes under the name of *amount collected*. This is sometimes called the *gross proceeds*.

Net proceeds means the amount that is left after all the expenses of collection have been taken out. In this case it is $38.

Gross proceeds means the whole amount collected. In this case it is $41.

(The items *net* and *gross* are in such common use as business terms that they should be carefully developed and worked into the child's vocabulary.)

The amount of a *debt* and the amount *collected* must be very carefully discriminated. Illustrative problems should be given in which the collector was able to collect only a part of the debt. (See *3* below.)

When the agent collects the full amount of the debt, the *debt* and the *amount collected* are equal. When the amount of the collection, the *gross proceeds*, is less than the full amount of the account, the pupil must discriminate in making up his statement "items."

The "items" in a collecting problem are

three—the gross proceeds (amount collected), the commission, and the net proceeds.

The *whole* in a collecting problem is the amount actually collected, the gross proceeds.

The term agent should be illustrated and defined. There are agents of all kinds—collecting agents, purchasing agents who buy goods for merchants or others, selling agents, insurance agents, etc. What these various agents do is a part of the development of the term.

Agents are sometimes employed at regular salaries. The problems of collecting have nothing to do with the pay of salaried agents. Their pay is not classed under commissions.

"Collecting" has to do with the work and compensation of agents who collect for some part of the amount collected.

These are some of the facts that must be taught. This "content" of collecting is the most valuable part of the work in the subject. We should regard the computation part as entirely secondary, as being for the sake of the content more than for the sake of arithmetical power. However, an informational outline in this subject that does not provide for simple computations would lack vitality.

2. *The Factors in Collecting*

The introduction should be made through a simple problem.

"A collector was paid $3 for collecting an account of $30."

"What items do we know in this problem?"

("The amount collected, $30; the commission, $3; and the net proceeds, $27.")

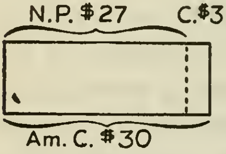
"How many items?" ("Three.")

"In a collecting problem, there are three and only three items. The amount collected is the *whole*, the other items are *parts*."

"You may write the fractions." ("$\frac{30}{30}$, $\frac{3}{30}$, $\frac{27}{30}$.")

The questions are now asked as to what each is the fraction of.

"In a collecting diagram should we 'add on' or should we 'take off'"?



"You may make the diagram."

The experience that the pupil has had in gain, loss, and trade discount, will make explanations almost unnecessary. The diagram will be made without suggestion.

("This is a collecting problem therefore we take off.")

"You may outline and solve."

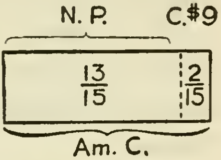
1 = Am. C. = $30	$\frac{30}{30} = 1$ = Amount Collected.
= C. = $ 3	$\frac{3}{30} = \frac{1}{10}$ = Commission.
= N. P. = $27	$\frac{27}{30} = \frac{9}{10}$ = Net Proceeds.

"A collector received \$9 for collecting a debt on a commission of $\frac{2}{15}$." "Diagram."

("This is a collecting problem. We take off.")

("The whole is the amount collected.")

$(\frac{15}{15}),\ 1 = \text{Am. C.} = \$67.50$
$\frac{2}{15} = \text{C.} = \$\ 9.00$
$\underline{\frac{13}{15} = \text{N. P.} = \$58.50}$
$\frac{2}{15} = \$9,\ \text{C.}$
$\frac{1}{15} = \frac{1}{2} \text{ of } \$9 = \$4.50$
$\frac{13}{15} = 13 \times \$4.50 = \$58.50,\ \text{N. P.}$
$\frac{15}{15} = 15 \times \$4.50 = \$67.50,\ \text{Am. C.}$



Several problems in collecting should now be given.

(No further suggestions need be made here on this subject of collecting. The plan as to the pupil's oral work in connection with the diagrams and outline statements, as to questions to be asked, and as to the forms of diagrams and statements, is the same as that for gain, loss, and trade discount.)

3. Special Problems on Amount Collected

The child is taught in connection with the informational work on this subject that the amount collected is not necessarily the face of the debt or account. No ordinary amount of instruction, however, will give this the force and the reality that it acquires when the fact comes to him when he is in difficulties. The following problem will illustrate:

"A collector was given an account of $80 for collection at a commission of $\frac{1}{9}$. As the result of his work he paid over to the merchant $56."

Most pupils, regardless of the fact that they may have solved a dozen or more collecting problems, will make the statement as follows:

$$\begin{array}{rll} (\frac{9}{9}),\ 1 = & \text{Amt. C.} & = \$80 \\ \frac{1}{9} = & \text{C.} & = \\ \frac{8}{9} = & \text{N. P.} & = \$56 \end{array}$$

The analytical computation will show the pupil that the statement will not prove. It is out of balance.

The pupil reaches the conclusion at once, the agent was dishonest.

With regard to questions of honesty, the teacher must state, and state with emphasis, that problems for solution assume *honesty* in transactions. This question must not enter into the considerations.

This being disposed of, the only allowable suggestion in the difficulty is to read the problem again to see what it says. Does it state any thing positively so that we may find what is wrong. The 80-dollar item or the 56-dollar item must be wrong.

This reading may lead to the conclusion that the merchant received \$56 and this is surely the *net proceeds.*

The pupils, if they so decide, make the statement with \$56. The teacher puts on the blackboard the statement with \$80 as follows:

The pupil's statement —

$$\begin{array}{lll}
(\frac{9}{9}),\ 1 & = \text{Amt. C.} & = \$63 \\
\frac{1}{9} & = \text{C.} & = \$\ 7 \\
\frac{8}{9} & = \text{N. P.} & = \$56 \\
\hline
\frac{8}{9} & = \$56 & \\
\frac{1}{9} & = \frac{1}{8} \text{ of } \$56 = \$7, \text{ C.} & \\
\frac{9}{9} & = 9 \times \$7 = \$63, \text{ Amt. C.} &
\end{array}$$

The teacher's statement–

$$\begin{array}{lll}
(\frac{9}{9}),\ 1 & = \text{Amt. C.} & = \$80 \\
\frac{1}{9} & = \text{C.} & = \$\ 8.89 \\
\frac{8}{9} & = \text{N. P.} & = \$71.11 \\
\hline
\frac{9}{9} & = \$80 & \\
\frac{1}{9} & = \frac{1}{9} \text{ of } \$80 = \$8.88\frac{8}{9}, \text{ C.} & \\
\frac{8}{9} & = 8 \times \$8.88\frac{8}{9} = \$71.11\frac{1}{9}, \text{ N. P.} &
\end{array}$$

Both statements balance.

The teacher's statement involves dishonesty or error in computation on the part of the agent. Either supposition is impossible to consider in school room work.

The pupil's statement must therefore be the correct one.

What is the conclusion? That the collector failed to collect the whole debt.

The problem purposely left this fact out.

The solution should discover it.

Problems of this kind should be given occasionally in the mixed problem work in *4*. They serve a very valuable purpose—

(1) In forcing the child to be discriminating in determining the "items" in a problem.

(2) In teaching him how to use a statement that does not balance—to check up the error.

4. Mixed Exercises for Seat Work

These should include problems in gain, loss, trade discount, and collecting. The problems in collecting should include occasionally one like those discussed in *3* above.

E. Commission

1. Selling Commission

a. THE VOCABULARY

The information work under this subject includes the following:

Commission merchant—a merchant who sells or buys goods for others, generally receiving pay for his services in proportion to the amount of the sales or purchases.

Commission merchants for arithmetical purposes are selling merchants and buying merchants.

One commission merchant may be both a selling agent and a buying agent.

Commission—the amount paid the agent for services in buying or in selling.

Amount sold—the amount paid the agent for the goods, the *gross proceeds* of the sale. This is called the selling price.

Net proceeds—the amount paid by the selling agent to the person for whom he sold the goods. It is the gross proceeds less the agent's commission for selling. It is what the merchant receives from his selling agent.

Kinds of commission. There are two kinds of commission, selling commission and buying commission.

b. The Factors in Selling Commission

The items in a selling commission problem are three—the *gross proceeds* (the agent's selling price), the *net proceeds*, and the *agent's commission.*

The *whole* in a selling commission problem—the *agent's* selling price (the gross proceeds).

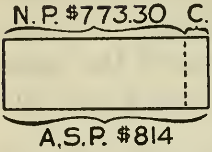
c. Diagrams and Statements

"A grain dealer sent wheat to his agent for sale. The grain was sold by the agent for $814. The agent sent the dealer $773.30."

"This is a problem in selling commission."

"The *whole* is the agent's selling price, $814."

"In a selling commission problem we take off."



$(\frac{20}{20}),\ 1 = \text{A. S. P.} = \$814.00$

$\frac{1}{20} = \text{C.} = \$\ 40.70$

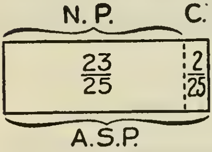
$\frac{19}{20} = \text{N. P.} = \$773.30$

$\frac{814}{814} = 1 = \text{A. S. P.}$

$\frac{40.7}{814} = \frac{1}{20} = \text{C.}$

$\frac{773.3}{814} = \frac{19}{20} = \text{N. P.}$

"An agent sold goods on a commission of $\frac{2}{25}$ and sent to the dealer as the proceeds of the sale $66."



$(\frac{25}{25}),\ 1 = \text{A. S. P.} = \$71.74$

$\frac{2}{25} = \text{C.} = \$\ 5.74$

$\frac{23}{25} = \text{N. P.} = \$66$

$\frac{23}{25} = \$66$

$\frac{1}{25} = \frac{1}{23}$ of $66 $= \$2.86\frac{22}{23}$

$\frac{2}{25} = 2 \times 2.86\frac{22}{23} = 5.73\frac{21}{23}$

$\frac{25}{25} = 25 \times 2.86\frac{22}{23} = 71.73\frac{21}{23}$, A. S. P.

d. Mixed Exercises

Several exercises should be given in this class of problems and then they should find a place in the daily seat work among the mixed problems.

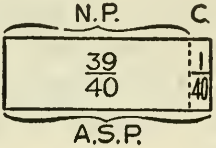
Note.—The following problem with its diagram and solution are given here to show the teacher the form of work with problems which involve charges for freight, insurance, etc.:

"Sent a consignment of goods to my agent for sale on a commission of $\frac{1}{40}$. The agent sold the goods and sent me, after deducting his commission and $12 for freight and cartage charges, $507."

Remittance by the agent	$507
Charges deducted by the agent	$ 12
Net Proceeds of Sale	$519

$1 = \text{A. S. P.} = \$532.31$
$\frac{1}{40} = \text{C.} = \$3.31$
$\frac{39}{40} = \text{N. P.} = \$519$

$\frac{39}{40} = \$519$
$\frac{1}{40} = \frac{1}{39}$ of $\$519 = \$13.30\frac{10}{13}$
$\frac{40}{40} = 40 \times \$13.30\ \frac{0}{3} = \$532.30\frac{10}{13}$, A. S. P.



The teacher will observe that the commission is regularly and always reckoned on *what the agent sells the goods for*. In the problem above the agent actually sold the goods for $532.31. He took for his commission $\frac{1}{40}$ of this. He paid the freight charges out of the remainder, $519. This left for the consignor $519 − $12 = $507. The fact of freight, insurance, or other charges makes no difference in the computation of commissions. This principle applies in "Buying Commission" work also. Commissions, net proceeds, etc., are not changed in form of reckoning on account of freight or other charges.

2. *Buying Commission*

The information connected with this subject is practically the same as for "selling commission" excepting that—

The *whole* is the *agent's buying price*, what the agent pays for the goods.

The agent adds his commission to what he pays for the goods, the total of these two items being the amount due the agent.

The *total cost* of the goods, therefore, is the agent's buying price and the commission.

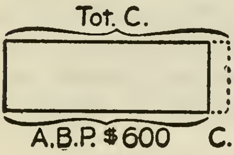
The "items" in a "buying commission" problem are three—the agent's buying price, the commission, and the total cost.

"A commission merchant bought goods for his principal for \$600. His commission was \$25."

"This is a 'buying commission' problem."

"The whole is the agent's buying price, \$600."

"It being a 'buying commission' problem, we add on."

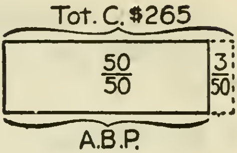


$$\begin{aligned} 1 &= \text{A. B. P.} &&= \$600 \\ \tfrac{1}{24} &= \text{C.} &&= \$\ 25 \\ \underline{\tfrac{25}{24}} &\underline{= \text{Tot. Cost}} &&\underline{= \$625} \end{aligned}$$

$$\begin{aligned} \tfrac{600}{600} &= 1 = \text{A. B. P.} \\ \tfrac{25}{600} &= \tfrac{1}{24} = \text{C.} \\ \tfrac{625}{600} &= \tfrac{25}{24} = \text{Tot. Cost.} \end{aligned}$$

"A merchant purchased goods through a commission house at a commission of $\frac{3}{50}$. The total cost of the goods was \$265."

$(\frac{50}{50}), 1 = \text{A. B. P.} = \$250$
$\frac{3}{50} = \text{C.} = \$\ 15$
$\frac{53}{50} = \text{Tot. Cost} = \$265$

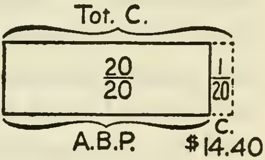


$\frac{53}{50} = \$265$
$\frac{1}{50} = \frac{1}{53}$ of $\$265 = \$5$
$\frac{3}{50} = 3 \times \$5 = \$15$, C.
$\frac{50}{50} = 50 \times \$5 = \$250$, A. B. P.

The following problem is given here with the diagram and solution in order to show the teacher the proper treatment of problems which involve expenditures by the agent for freight, insurance, etc.

"My agent bought goods for me on a commission of $\frac{1}{20}$. His commission was $14.40 and there were expenses as follows: freight $4.30, insurance $2.60."

$(\frac{20}{20}), 1 = \text{A. B. P.} = \$288.00$
$\frac{1}{20} = \text{C.} = \$\ 14.40$
$\frac{21}{20} = \text{Tot. Cost} = \$302.40$



$\frac{1}{20} = \$14.40$
$\frac{20}{20} = 20 \times \$14.40 = \$288$, A. B. P.
$\frac{21}{20} = 21 \times \$14.40 = \$302.40$, Tot. Cost

Total Buying Cost.		$302.40
Charges—Freight . . .	$4.30	
Insurance.	$2.60	6.90
Cost of goods to merchants		$309.30

MIXED EXERCISES

Buying Commission problems should now be among the mixed exercises for daily seat work.

F. Fire Insurance

1. The Plan

Every pupil of the age to take up applied percentage in school is more or less familiar with the many forms of insurance that touch ordinary life. Fire insurance, life insurance, accident insurance, burglar insurance, automobile insurance, etc. are matters that could hardly be unfamiliar terms with children who have had any contact with every-day affairs. They know enough of fire insurance to make it unnecessary to make any developments. It is perfectly proper for the teacher to assume that policy, premium, insurer, and insured are well enough understood for her to introduce fire insurance instruction by simply organizing and making more definite the knowledge that the pupils possess.

2. Oral Exercises—Introduction

This introduction should be conducted in *oral* exercises devoted wholly to discussions of what *they know* and *think* about fire insurance with a view to correct any erroneous impressions in minor matters that some may have, but particularly to make definite their knowledge of the meanings of terms.

The matters to be discussed in these exercises should cover the following points: *

(1) That the fire insurance company promises to make the owner sure (to protect him) against loss by fire, to a given amount.

(2) That the owner pays the company for assuming this risk—*premium.*

(3) That the promise must be written so that there could be no dispute later about the agreement. This brings *policy*, the written promise, into use and definition.

(4) That this promise relates to protection from loss on particular property, which must be spoken of in the promise very definitely.

(5) That the promise must relate to such protection for a definite period of time, the agreement being very clear as to when the protection begins and when it ends—expires. *Term* of the policy, *expiration.*

(6) That the insurer promises protection to the *insured.*

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* The teacher will understand that although for the sake of brevity of statement, it has been necessary to make this outline of "points" very formal, the oral work which it suggests must be conducted in a manner as informal as possible. It has not seemed necessary to suggest the method—simply the material

3. Written Exercises

a. The Policy

Following the oral introduction, the teacher should take up written construction work on the insurance policy. This consists in writing the insurance agreements in a rough form—mere written promises that cover the "points" which the oral introduction has organized and emphasized. It is a series of lessons in written language in which the aim is written promises to protect against loss, the requirement being that the pupil state the full promise in definite, connected, clear language.

Note.—The teacher is warned that a real policy of fire insurance must not under any circumstances be brought before the class until *after* the *completion* of this written work. The study of a real policy would destroy the value of the written work here contemplated. The child would try to copy what he might hastily glean from looking at or discussing such policy. What we want here is concentration upon the essential points in an insurance promise. Stating his conception of it makes for the closest attention to these points without the form or language being influenced by what the child may have seen on a formal policy. The real insurance policy is a document that only the mature, trained mind can understand. Every child should have

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of the discussion. These discussions should be kept, so far as possible, from extending beyond the simple facts and terms outlined.

an opportunity to see one, but it should be *after* he has written policies of his own in his own language stating the essential points of the agreements, the real policy being studied very briefly afterwards with the sole purpose of picking out from the verbiage the fact that the policy contains in long drawn out form, so far as the promises and period and signature are concerned, the very points that he has forecast in his own little policies. What the child should know about fire insurance policies is what they are all about. This he gets from writing his own insurance policies. The use of the real policy is to verify his forecast.

b. The Essential Features

So far as the elementary school pupil is concerned, the essential features of a fire insurance policy—the features that he must embody in the written policies that he composes—are—

(1) The date of the promise, like the heading of letters that he has been taught.

(2) The receipt for the premium.

(3) The promise to protect.

(4) The amount of the promised protection—the face of the policy.

(5) The definite statement of the property insured.

(6) The date when the protection begins.

(7) The date when the protection ceases.

(8) The name of the person insured

(9) The signature of the insurer.

c. PRELIMINARY INFORMATION

This outline should be written on the blackboard before the pupils begin to write their policies (promises).

The points in the outline have been covered in the Oral Exercises (see *2* above). The outline (in *b* above) is placed on the blackboard now as a guide in writing—to know when the "promise" (his *policy*) is complete.

The pupils should know in advance that the promise must, like any letter, have a regular "heading" but *not* a "salutation."

The letter should begin with a receipt for the premium.

Before the pupils are ready for this work, they should know how to describe houses and lots.

(In some cities lots are described "by metes and bounds." In such cases the pupils' "promises" (which must always relate to houses or other property within view from the school windows or to houses whose locations and descriptions are well known) should describe houses by *street* and *number*. In cities where descriptions are in terms of Lot—of Block—in ——— Addition to the City of ———, or similar plan, this form is sufficiently simple. We should aim for descriptions that are as simple as is compatible with definiteness of location so far as the child knows. Our business now is the insurance policy, not real-estate descriptions.

Pupils must also be taught to describe houses in a very simple way—one story, frame, slate roof, dwelling house; two story, brick, shingle roof, dwelling; three story, brick, gravel roof, store building; etc.

It is well for the teacher to know in advance approximately what the annual insurance rates are so that she may guide the pupils against putting premium rates ridiculously high or low. If thought desirable, premium rates and consequently premiums may enter into the Oral Exercise work above. In that case pupils would be prepared in this matter at this time.)

d. THE "PROMISES"

The insurer should, at first, be the pupil himself. The insured should be the owner of the house, which should be some particular house in the neighborhood. The amount of the insurance—in each case the face of the policy—should be agreed upon in advance. The amount of the premium should also be agreed upon.

Later, the insurer should be some person in the room other than the writer. The aim here is to give experience in having the right name signed to the promise. The pupil needs this experience to compel him to discriminate as to whose signature the "promise" requires.

Last of all, the "promises" are to be made by regular insurance companies. Every child knows the names of half a dozen fire insurance companies. These should furnish the names of the insurers in their "promises."

The first set of compositions ("promises") should be corrected first as language exercises —corrected as to spelling, English, and so much of punctuation and capitals as their language course has thus far developed. This correction work ought to be in the form of class exercises with several of the papers copied on the blackboard and not by the teacher after school hours. The pupils should have a part in the correcting.

After the compositions on the blackboard have been corrected thus as to the English, etc., the teacher should suggest any omissions that there may be in each—

"You did not receipt for the premium"; or,

"You do not have a signature. A person does not *promise* until he *signs*. An unsigned promise is not a promise"; or,

"You did not state the time of expiration"; etc.

The compositions are now laid aside and each writes another "promise." The English is again corrected and the omissions again noted. After three or four "promises" have been written, complete productions will be found. These repeated compositions will be valuable for written English and will fix thoroughly the essential points in a fire policy.

The following has been selected as a good representation of the form of the completed "promise" that this written work contemplates:

Chicopee, Mass.,
February 8, 1917.

I have received from N. P. Carter seven dollars and fifty cents. For this I promise to insure him against loss by fire on his two-story frame, slate roof dwelling, No. 256 Grove Street, Chicopee Falls, to the amount of one thousand two hundred dollars, from twelve o'clock noon February 9, 1917, to twelve o'clock noon February 9, 1918.

ALLEN T. HOWARD.

e. THE OWNER

After each written exercise, the pupil should be asked the questions—

Who owns that promise (policy)?

In whose pocket or box or safe should it be placed?

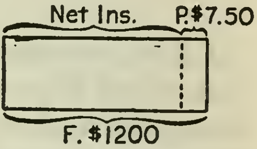
Why should he have it?

It is with these "promises" as with receipts, orders, notes, etc. The pupil must *learn* who the proper holder is. Without this feature of the work, the teacher may find the pupil delivering a policy to the *insurer*.

The above questions should be asked of each written promise whether it is perfect or imperfect.

f. Diagrams and Solutions

The "items" in insurance are three—*premium*, *face*, and *net insurance*—the net return in case of total loss. The Carter "policy" above would appear as follows:



$$\begin{array}{rl} 1 & = \text{F.} = \$1200 \\ \frac{1}{160} & = \text{P.} = \$7.50 \\ \frac{159}{160} & = \text{N. Ins.} = \$1192.50 \\ \hline \frac{1200}{1200} & = 1 \quad = \text{Face} \\ \frac{7.50}{1200} & = \frac{1}{160} = \text{Rate of Premium} \\ \frac{1192.50}{1200} & = \frac{159}{160} = \text{Rate of Net Insurance.} \end{array}$$

The pupil's oral work in solution is—

("This is an insurance problem. The whole is the face of the policy."

"In an insurance problem we take off."

"I make a diagram. The whole is the face. The net insurance and the premium are parts.")

Many exercises should be given in diagramming, in solutions, and in writing "promises" from stated conditions.

g. The Real Insurance Policy

A regular fire insurance policy is now in order. An *expired* policy is as useful as one that is alive and it is easier to obtain.

The use to be made of it is to find in it the "points" that the pupils' "promises" have contained. It is to verify their work. The other matter in the policy may or may not be noted as the teacher may think best. The "points" are the essential and only essential features.

SUMMARY.

LOSS AND GAIN

The factors in loss and gain are the cost of the goods, the gain or the loss, and the selling price. The cost is the whole; the other two factors, parts.

Every statement in loss and gain must be made with a view to computing all the facts in the transaction—the items and the fractions which express the measures of the relations.

The pupil is taught to show the factors and the fractions of each problem in diagram form.

TRADE DISCOUNT

The factors in trade discount are the marked price, the discount, and the selling price. The marked price is the whole; the other factors, parts.

The pupil learns to show the factors in their relations with diagrams.

Double trade discounts are developed with the forms of their diagrams and statements.

COLLECTING

The factors in collecting are the amount collected, the commission, and the net proceeds.

The forms of statements and the diagrams are on the same plan as those for loss and gain and trade discount.

COMMISSION

This subject includes selling commission and buying commission. The factors in selling commission are agent's selling price, the commission, and net proceeds; in buying commission—agent's buying price, commission, and total cost to the merchant.

The plan for commission follows that for the other subjects.

FIRE INSURANCE

The factors are the amount insured, the premium, and the net insurance.

The statement and diagram plans follow those for the other subjects.

The development aims to teach the pupil in simple form the essential features of a fire insurance policy.

CHAPTER III

INTRODUCTION OF DECIMALS—100THS

Topics:—Decimal Fractions in Applied Problems. Taxation Terms and Solutions. The Methods with Multiple Problems

A. Decimal Fraction Reviews

Before the introduction of decimals is attempted, the teacher must know that her pupils are fresh in the following work with decimal fractions:

(1) *Making* and *reading* the *series* of decimal fraction equivalents from .7, .9, .04, .13, .5, .15, 1.01, 2.2, 5.3, etc. (See Decimal Fractions, Chapter VII, *B* and *C*.) This includes the reading of decimal mixed numbers as mixed numbers or as improper fractions as the teacher shall direct. (See Decimal Fractions, Chapter VI, *C*.)

(2) *Writing* decimal fractions in common fraction form and in decimal fraction form as the teacher may direct. The dictations are oral.

(3) Reading hundredths in per cent language.

(4) Translating oral per cent dictations into written decimal form.

(5) Giving readily the common fraction equivalents of per cent expressions.

B. Per Cent Introduced in Solutions

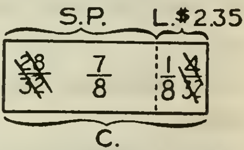
1. The Plan—Two or More Fractions

The aim in this step is to teach the child that the loss and gain fractions—C., G., and S. P.; and C., L., and S. P., may be in any case fractions of any name that the individual may choose. Take for illustration the problem, Sold at loss of $\frac{1}{8}$ and lost $2.35. The fractions may be 8ths or 16ths or 64ths or 80ths or 100ths as one may elect.

More than that, we may have two sets of fractions or three sets if we choose. In the analytical solution we may use any one of the sets.

$\frac{32}{32}, \frac{16}{16}, (\frac{8}{8}), 1 = \text{C.} = \$18.80$
$\frac{4}{32}, \frac{2}{16}, \frac{1}{8} = \text{L.} = \$\ 2.35$
$\frac{28}{32}, \frac{14}{16}, \frac{7}{8} = \text{S. P.} = \$16.45$
$\frac{4}{32}{}^{*} = \frac{1}{8} = \$2.35$ L.
$\frac{28}{32} = \frac{7}{8} = 7 \times \$2.35 = \$16.45$, S. P.
$\frac{32}{32} = \frac{8}{8} = 8 \times \$2.35 = \$18.80$, C.

The diagram would show the same proportional parts whatever fraction name is used. In the problem above the diagram would be—



If the fractions had been 72ds or 100ths, the fractions entered on the diagram would have been $\frac{1}{8}$ and $\frac{7}{8}$ or .12½ and .87½.

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* One point of *emphasis* in all solutions is that we must use the fractions in their lowest terms.

2. *The Introduction—Fractions in Lowest Terms*

a. Two Illustrative Problems

No. 1. "Sold goods for \$28 and made a profit of $\frac{1}{4}$."

"Use 20ths."

$\frac{20}{20}$, ($\frac{4}{4}$), 1 = C. = \$22.40
$\frac{5}{20}$, $\frac{1}{4}$ = G. = \$5.60
$\frac{25}{20}$, $\frac{5}{4}$ = S. P. = \$28

$\frac{25}{20} = \frac{5}{4}$ = \$28, S. P.
$\frac{5}{20} = \frac{1}{4} = \frac{1}{5}$ of \$28 = \$5.60, G.
$\frac{20}{20} = \frac{4}{4}$ = 4 × \$5.60 = \$22.40, C.

No. 2. "Sold goods at a profit of $\frac{2}{5}$ and gained \$7.60."

"Use 100ths."

1.00, ($\frac{5}{5}$), 1 = C. = \$19
.40, $\frac{2}{5}$ = G. = \$7.60
1.40, $\frac{7}{5}$ = S. P. = \$26.60

.40 = .4 = \$7.60, G.
.1 = $\frac{1}{4}$ of \$7.60 = \$1.90
1.00 = 1.0 = 10 × \$1.90 = \$19, C.
1.40 = 1.4 = 14 × 1.90 = \$26.60, S. P.

b. Common Fraction Solutions in Lowest Terms

No 1 of the above solutions shows a common fraction rate of gain ($\frac{1}{4}$) written with two fractions and solved, as directed, in 20ths.

The only teaching necessary is (1) with regard to the way to write the second fraction

(20ths) at the left of the statement, and (2) that the solution must be made by the use of the $\frac{20}{20}$, $\frac{5}{20}$, and $\frac{25}{20}$ in their lowest terms.

Several exercises using loss or gain or trade discount or collecting, etc. should be given for solution, using common fractions as here and requiring the use of fractions not in their lowest denominators. The solutions will bring out that the analytical work must be done with these fractions reduced to their lowest terms.

c. DECIMAL FRACTION SOLUTIONS IN LOWEST TERMS

(1) Regular Decimal Language—Hundredths.

No. 2 of the solutions asks for the use of 100ths. In the use of 100ths the fractions must be written *invariably* in the *decimal form*. Do not use 100ths in common fraction form—denominators expressed.

The same principle applies as to using the fraction in the solution in its lowest terms. The fractions here are hundredths. Their lowest *decimal* terms are 10ths. We can not express the three with a common *decimal* denominator lower than 10ths. The fractions before reduction are 40 hundredths, 100 hundredths, and 140 hundredths. They should be so read. After reduction they are 4 tenths 1 tenth, 10 tenths, and 14 tenths.

Several exercises, using problems in any or all the applied percentage subjects that have been developed, should now be given requiring the use of 100ths. The problems should state the rates of the loss, gain, discount, etc., in common fractions as in No. 2. These fractions should be 5ths, 10ths, and halves. These fractions are suggested because they will involve reductions to lowest *decimal* terms.

(2) Language of Per Cent—Two Fraction Outlines.

Problems should now be given in *per cent* language.

The *outlines* should have both common and *decimal* fractions.

The *solutions* should be made with the *decimal* fractions.

The *per cent* should be written *in all cases* with the use of the *decimal point*. In outlines, 10%, 20%, etc. must be written .10, .20, etc.

Complex fraction per cents—$33\frac{1}{3}$%, $12\frac{1}{2}$%, etc.—should not be used while the analytical work is being developed. They must be used later but the solutions with such fractions will be by translating them to common fraction form.

A collector received $13.90 for collecting at 10%.

$$\begin{array}{llll}
1.00,\ \left(\frac{10}{10}\right),\ 1 & = \text{Amt. C.} & = \$139. \\
.10,\ \frac{1}{10} & = \text{C.} & = \$\ 13.90 \\
.90,\ \frac{9}{10} & = \text{N. P.} & = \$125.10 \\
\hline
\end{array}$$

$$\begin{aligned}
.10 &= .1 = 13.90,\ \text{C.} \\
.90 &= .9 = 9 \times \$13.90 = \$125.10,\ \text{N. P.} \\
1.00 &= 1.0 = 10 \times \$13.90 = \$139.,\ \text{Amt. C.}
\end{aligned}$$

Gave a discount of 30% and sold for $129.

$$\begin{array}{llll}
1.00,\ \left(\frac{10}{10}\right),\ 1 & = \text{M. P.} & = \$184.29 \\
.30,\ \frac{3}{10} & = \text{D.} & = \$\ 55.29 \\
.70,\ \frac{7}{10} & = \text{S. P.} & = \$129 \\
\hline
\end{array}$$

$$\begin{aligned}
.70 &= .7 = \$129,\ \text{S. P.} \\
& \ .1 = \tfrac{1}{7} \text{ of } \$129 = \$18.42\tfrac{6}{7} \\
.30 &= .3 = 3 \times \$18.42\tfrac{6}{7} = \$55.28\tfrac{4}{7},\ \text{D.} \\
1.00 &= 1.0 = 10 \times \$18.42\tfrac{6}{7} = \$184.28\tfrac{4}{7},\ \text{M. P.}
\end{aligned}$$

A commission merchant received $39.25 for selling goods on a commission of 2%. He paid out $3.50 for freight.

$$\begin{array}{llll}
1.00,\ \left(\frac{50}{50}\right),\ 1 & = \text{A. S. P.} & = \$1962.50 \\
.02,\ \frac{1}{50} & = \text{C.} & = \$\ \ 39.25 \\
.98,\ \frac{49}{50} & = \text{N. P.} & = \$1923.25 \\
\hline
\end{array}$$

$$\begin{aligned}
.02 &= 39.25,\ \text{C.} \\
.01 &= \tfrac{1}{2} \text{ of } \$39.25 = \$19.625 \\
.98 &= 98 \times \$19.625 = \$1923.25,\ \text{N. P.} \\
1.00 &= 100 \times \$19.625 = \$1962.50,\ \text{A. S. P.}
\end{aligned}$$

Net Proceeds of Sale......	$1923.25
Freight....................	$3.50
Due to consignor	$1919.75

The above outlines and analytical solutions will be sufficient suggestions as to the introductory work with fractions in per cent language. The two-fraction outlines should be continued for several weeks. The common fractions serve as a check against errors in the decimal fractions. The child should have this common fraction as a means of verifying the decimal.

Note—The teacher will observe that in all the problems given in this outline for illustration purposes the usual questions with which book problems conclude, What was the gain? or What was the rate of commission? or What was the selling price? or What was the premium? etc., have been omitted. The problems are simply *stated* and the pupil is left to find all the results.

This is the form in which all problems in percentage prepared by the teacher for development and practice purposes should be given. The pupil should acquire the habit of finding *all* the results in each problem. This should be done even when the question, What is the —? is asked. This full solution is a great check against error. The student who always completes the outline statement will have less errors in reasoning and computations.

When the *question* is asked with the problem, the pupil should complete the outline and then write beneath his outline and solution the answer to the question—

The rate of loss was ——; or,

The commission was —— dollars; or other proper answer.

(3) Mixed Exercises. Many miscellaneous problems in loss, gain, trade discount, collecting, etc., for practice in solutions in per cent language should be given.

C. Taxation

This is a subject that involves a wide range of information that ordinarily is a part of the regular course in civics. It includes information relating to the need of money for public purposes; the sources of the funds; and the methods of apportionment, collection, and expenditure. It is introduced here because of the need of *decimals* and *per cent* in its development.

The vocabulary includes the common technical terms—real estate, personal property, assessor, collector, poll tax, property tax, direct tax, indirect tax, income tax, duty, tariff, import, export, impost, ad valorem, specific, excise, internal revenue, stamps, etc.

Property taxes, and all duties and internal revenue taxes that are ad valorem are included properly under the percentage applications.

Property-tax outline statements and solutions are of two kinds—(1), those that have to do with tax rates and *total amounts* of taxes; and (2), those that relate to the taxes of *individual taxpayers*. The two following outlines and solutions illustrate the two kinds:

(*a*) 1.00 = Taxable Property = $2150000
.01½ = Tax. = $32250

$\frac{2150000}{2150000} = 1.00 = \text{Whole.}$

$\frac{32250}{2150000} = .015 = \text{Rate.}$

(*b*) 1.00 = Taxable Property = $2850
.01½ = Tax. = $42.75
1.00 = $2850

$.01 = \frac{1}{100} \text{ of } \$2850 = \$28.50.$

.01½ = 1½ × $28.50 = $42.75, Tax.

Of the two outlines and solutions above—

(*a*) is the assessors' first problem, to find the town tax-rate. The problem outlined is that of finding this rate when the total real and personal property subject to taxation in the town is $2150000 and the amount to be raised by the property tax is $32250. It shows a tax-rate of 1½ per cent.

(*b*) is one of the assessors' second problems, that of finding the amount of property tax to be paid by the individual taxpayers. In this case, the problem is that of the amount to be paid by an individual whose taxable real estate and personal property is $2850.

It will be observed that in taxation problems which come under percentage there is in each a *whole* and *one* "part." The *outlines* and solutions for indirect tax problems (ad valorem) need not be shown here. They are similar to those for property taxes.

D. Multiple Problems in Percentage

Problems which are included under this head are those in which two or more solutions are involved in each. Illustrations of this class of problems are afforded by trade discounts in which there are two or three successive discounts from an asking price.

Whether or not such problems should be given in a minimum course in arithmetic or should be reserved for advanced courses for commercial students is not a question that a teaching outline is called upon to discuss. Such problems arise in business computations and are given in reputable textbooks. An outline of this kind should, therefore, provide for their development by those who wish to include them in percentage training.

1. Diagrams and Oral Work

The Diagram Plan.

The following multiple problem in trade discount and gain will be sufficient to show the diagram plan for all multiple problems:

Sold goods at a discount of 25 per cent and made a profit of 10 per cent. The selling price was $39.60.

NOTE 1.—The pupil will observe when he reads this problem that it contains a reference to trade discount and a reference to a profit on the sale. He *must*, therefore, prepare to solve *two* problems. This suggestion applies to *all* multiple problems in percentage. There are as many separate distinct wholes, diagrams, outlines, and solutions in a problem as there are percentage applications involved. If there are four successive discounts in a problem, there are four wholes, four diagrams, four outlines, and four solutions.

("This is two problems, one in trade discount and one in gain."

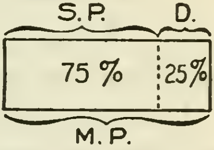
"The wholes are the marked price and the cost."

"I make the two diagrams."

"In the trade discount problem we take off. In a gain problem we *add on*.")

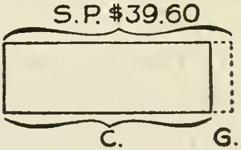
1.00, ($\frac{4}{4}$), 1 = M. P. = $52.80
.25, $\frac{1}{4}$ = D. = $13.20
.75, $\frac{3}{4}$ = S. P. = $39.60

.75 = $39.60, S. P.
.01 = $\frac{1}{75}$ of $39.60 = $.528
.25 = 25 × $.528 = $13.20, D.
1.00 = 100 × $.528 = $52.80, M. P.



1.00, ($\frac{10}{10}$), 1 = C. = $36
.10, $\frac{1}{10}$ = G. = $3.60
1.10, $\frac{11}{10}$ = S. P. = $39.60

1.10 = 1.1 = $39.60, S. P.
.10 = .1 = $\frac{1}{11}$ of $39.60 = $3.60, G.
1.00 = 1.0 = 10 × $3.60 = $36, C.



NOTE 2.—The teacher's attention is called to these two diagrams in this respect—that no attempt is made

to make the two proportional. The selling price on the first diagram is quantitatively equal to the selling price in the second. The diagram is not for solution purposes but to express the *relations* of parts to wholes. These diagrams serve their purposes if the relations of the trade discount "parts" (D. and S. P.) to their "whole" (M. P.) are shown, and the relations of the gain "parts" (S. P. and G.) to their "whole" (C) are shown. That one of these "wholes" is larger or smaller than the other is not a matter for the diagrams to show. *Each diagram* in an *applied percentage multiple problem is complete in itself*, having *no quantitative* or other *relation* to *any other diagram*. Each diagram is constructed as if there were no other in the problem.

2. *The Outline Statement and Solution Plan*

The following problem is illustrative of problems that have three or four solutions:

Marked goods to be sold at a certain profit but gave a discount of 20 per cent. The account had to be collected through an agency at a commission of 5 per cent. The agent's commission was $16.60. There was a loss of $1\frac{1}{5}\%$.

("This is three problems—one in trade discount, one in gain, and one in collecting."

"The wholes are marked price, cost, and amount collected."

"In a trade discount problem we take off, in a gain problem we add on, in a collecting problem we take off.")

$$\begin{array}{llll}
1.00,\ (\frac{5}{5}),\ 1 & = \text{M. P.} & = \$415. & \\
.20,\ \frac{1}{5} & = \text{D.} & = \$83. & \\
.80,\ \frac{4}{5} & = \text{S. P.} & = \$332. & \\
\hline
.80 = & .8 = \$332.\ \text{S. P.} & & \\
 & .1 = \frac{1}{8} \text{ of } \$332 & = \$41.50. & \\
.20 = & .2 = 2 \times \$41.50 & = \$83,\ \text{D.} & \\
1.00 = & 1.0 = 10 \times \$41.50 & = \$415,\ \text{M. P.} &
\end{array}$$

$$\begin{array}{lll}
1.00,\ (\frac{1000}{1000}),\ 1 & = \text{C.} & = \$319.23. \\
.01\frac{1}{5},\ \frac{12}{1000} & = \text{L.} & = \$3.83 \\
.98\frac{4}{5},\ \frac{988}{1000} & = \text{S. P.} & = \$315.40 \\
\hline
.988 = \$315.40,\ \text{real S. P.} & \text{(Owner's C. P.)} & \\
.001 = \frac{1}{988} \text{ of } \$315.40 & = \$.31923\frac{1}{13}. & \\
.012 = 12 \times \$.31923 & = \$3.83,\ \text{L.} & \\
1.000 = 1000 \times \$.3192 & = \$319.23,\ \text{C.} &
\end{array}$$

$$\begin{array}{lll}
1.00,\ (\frac{20}{20}),\ 1 & = \text{Amt. C.} & = \$332. \\
.05,\ \frac{1}{20} & = \text{C.} & = \$\ 16.60 \\
.95,\ \frac{19}{20} & = \text{N. P.} & = \$315.40 \\
\hline
.05 = \$16.60,\ \text{C.} & & \\
.01 = \frac{1}{5} \text{ of } \$16.60 = \$3.32. & & \\
.95 = 95 \times \$3.32 & = \$315.40,\ \text{N. P.}^{*} & \\
1.00 = 100 \times \$3.32 & = \$332,\ \text{Amt. C.}^{\dagger} &
\end{array}$$

This solution shows that the goods cost $319.23, that they were marked at $415, that they were actually sold for $332, that the collector collected the full amount, and that the

---

*This *real* S. P. (the owners).

†The "marked price" S. P.

net proceeds to the merchant (the collector's N. P.) was	$315.40
The solution brings out that—	
The goods cost	$319.23
The actual cash returns, the *real* S.P. (the owner's S. P.) was	$315.40
The actual loss	$ 3.83

3. The Solutions when there are Freight and Other Charges

The following problems show the form of solution of a two-part multiple problem in commission where there are items of freight, etc. to be considered.

A dealer sent potatoes to a commission merchant for sale at a commission of 4 per cent. The agent paid the freight and charges for handling on the potatoes amounting to $23. He was directed to invest the proceeds in sugar at a commission of 2%. This commission was $18.60. He paid freight and other charges on the sugar to the amount of $8.

("This is two problems, selling commission and buying commission."

"The wholes are the agent's selling price and the agent's buying price."

"In a selling commission we take off, in a buying commission we add on.")

$$\left.\begin{array}{llll}
1.00,\ (\frac{25}{25}),\ 1 & = \text{A. S. P.} & = \$1020.42 \\
.04,\ \frac{1}{25} & = \text{C.} & = \$\ \ 40.82 \\
.96,\ \frac{24}{25} & = \text{N. P.} & = \$\ 979.60 \\
\hline
\multicolumn{4}{l}{.96 = \$979.60,\ \text{N. P.}} \\
\multicolumn{4}{l}{.01 = \frac{1}{96} \text{ of } \$979.60 = \$10.204\tfrac{1}{6}} \\
\multicolumn{4}{l}{.04 = 4 \times 10.204\tfrac{1}{6} = \$40.816\tfrac{2}{3},\ \text{C.}} \\
\multicolumn{4}{l}{1.00 = 100 \times 10.204\tfrac{1}{6} = 1020.41\tfrac{2}{3},\ \text{A. S. P.}}
\end{array}\right\}\text{(c)}$$

$$\left.\begin{array}{llll}
1.00,\ (\frac{50}{50}),\ 1 & = \text{A. B. P.} & = \$930. \\
.02,\ \frac{1}{50} & = \text{C.} & = \$\ \ 18.60 \\
1.02,\ \frac{51}{50} & = \text{Tot. C.} & = \$948.60 \\
\hline
\multicolumn{4}{l}{.02 = \$18.60} \\
\multicolumn{4}{l}{.01 = \tfrac{1}{2} \text{ of } \$18.60 = \$9.30} \\
\multicolumn{4}{l}{1.00 = 100 \times 9.30 = \$930,\ \text{A. B. P.}} \\
\multicolumn{4}{l}{1.02 = 102 \times \$9.30 = \$948.60,\ \text{Tot. C.}}
\end{array}\right\}\text{(a*)}$$

$$\left.\begin{array}{lr}
\text{Total cost (to the agent) of the sugar} \ldots\ldots & \$948.60 \\
\text{Freight and other charges on the sugar} \ldots\ldots & \$8. \\
\text{Freight and other charges on the potatoes} \ldots\ldots & \$23. \\
\hline
\text{Total ``net proceeds'' from sale of the potatoes} \ldots\ldots & \$979.60
\end{array}\right\}\text{(b)}$$

4. Solutions by Reducing Fractions to Single Bases—Special

The following shows form of outline statement and solution in a trade discount problem of three discounts when the method is by reduction to a *single* selling-price fraction basis:

---

*(a), (b) and (c) show the solution steps in order.

Sold goods for $56.45 after discounts of 20, 10, and 2 per cent.

1.00 = M. P. = 1.00 = $80
*(a) .20 = D. = .20
.80 = S. P. = .80
.7056 = $56.45
(d) .01 = $\frac{1}{7056}$ of \$56.45 = \$.80
1.00 = 100 × $.80 = 80 M. P. (1st)

1.00 = M. P. = .80 = $64
(b) .10 = D. = .08
.90 = S. P. = .72
(e) .80 of $80 = $64, M. P. (2d)

1.00 = M. P. = .72 = $57.60
(c) .02 = D. = .0144 =
.98 = S. P. = .7056 = $56.45
(f) .72 of $80 = $57.60, M. P. (3d)

In this outline statement the fractions of the 2d and 3d discounts are reduced to the basis of the fractions of the 1st.

5. *Solutions with Fractions on Single Basis—Regular*

The following problem, valuable principally as an exercise to test the power to analyze, is given here to show a somewhat different form of reduction. In (4) the reduction is direct, the S. P. of one part being the M. P. of the next. The problem given below is more of a regular percentage problem.

---

* (a), (b), and (c) are the outlines; (d), (e), and (f) the solutions—in their order.

A commission merchant sold goods on a commission of 2 per cent and invested the proceeds in other goods on a commission of 3 percent. The combined commissions amounted to $31.55.

$$\left.\begin{array}{llll} 1.00,\ (\tfrac{50}{50}),\ 1 & = \text{A. S. P.} & = 1.00^{*} = \$650 & \text{(e)} \\ .02,\ \tfrac{1}{50} & = \text{C.} & = \ .02^{*} = \$\ \ 13 & \text{(f)} \\ .98,\ \tfrac{49}{50} & = \text{N. P.} & = \ .98^{*} = \$637 & \text{(g)} \end{array}\right\}\text{(a)}^{\dagger}$$

$$\left.\begin{array}{llll} 1.00,\ (\tfrac{100}{100}),\ 1 & = \text{A.B.P.} & = .95\tfrac{15}{103}^{*} = \$618.45 & \text{(i)} \\ .03,\ \tfrac{3}{100} & = \text{C.} & = .02\tfrac{88}{103}^{*} = \$\ \ 18.55 & \text{(j)} \\ 1.03,\ \tfrac{103}{100} & = \text{Tot. C.} & = .98^{*} = \$637 & \text{(h)} \end{array}\right\}\text{(b)}$$

$$\left.\begin{array}{ll} \text{Selling commission (A.S.P. basis)} & .02 \\ \text{Buying commission (A.S.P. basis)} & \underline{.02\tfrac{88}{103}} \\ \qquad \text{Total} \ldots\ldots\ldots\ldots\ldots\ldots & .04\tfrac{88}{103} \end{array}\right\}\text{(c)}$$

$$\left.\begin{array}{r} .04\tfrac{88}{103} = \$31.55 \text{ Tot. Com.} \\ .01 = \tfrac{1}{4\frac{88}{103}}(\tfrac{103}{500}) \text{ of } \$31.55 = \$6.50 \\ 1.00 = 100 \times \$6.50 = \$650 \text{ A. S. P.} \end{array}\right\}\text{(d)}$$

SUMMARY.

REVIEWS IN DECIMALS

The pupil is given special reviews of decimal series work also in reading and writing decimal proper and improper fractions in hundredths. This prepares for the use of decimal fractions in lowest terms.

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*These are the A. S. P. and the A. B. P. "rates" *all* reckoned on the A. S. P. "whole" (100%). It shows that the A. B. P. rate of C. was $2\frac{88}{103}\%$ of A.S.P. We may now add (c) these C. rates because they are fractions of the same kind of whole.

† (*a*), (*b*) and (*c*) show the outlines, solutions, and final results in order.

DECIMAL FRACTION SOLUTIONS

The chapter aims to teach the pupil that he may use a decimal fraction or any denomination of common fractions that he may choose in a given percentage solution but that, whatever the fraction used, the solution must be made with this fraction in its lowest terms. The decimal fraction solutions are introduced and the pupil taught to make them analytically.

TAXATION

Taxation, an application of decimals in "percentage," has two distinct lines of computation, each with *two* factors. The two lines of work in this subject are—

(1) Finding a town or city tax rate; and (2) Finding the tax of an individual taxpayer.

The *whole* in the former is the total of the taxable property in the town or city. The part is the amount to be raised by direct property tax.

The *whole* in the latter is the assessed value of the taxpayer's property subject to the tax.

Multiple Problems with Decimals.

Outlines of this class of problems are given for teachers who may wish to use them. The outline includes problems of the kind in which there are freight and other charges and also problems in which the *rates* must be reduced to a single basis.

CHAPTER IV

BASE, PERCENTAGE, AMOUNT, DIFFERENCE—PERCENTAGE CONCEPTS

Topic:—"General" Percentage—with Base, Rate and Percentage, Amount, and Difference

A. The Aim

All solutions so far have been what we may call *direct*. Loss problems were solved under loss terms; trade discount problems, under marked price, discount, and selling price; collecting problems, under the collection language of amount collected, commission, and net proceeds.

This is the way to solve such problems practically; but there is reason in saying that the *indirect* language—that of base, rate, percentage, amount, and difference—is too widely used and too serviceable at times to be omitted from a course of study in number that is not a *minimum*. This chapter is to provide an outline for those who wish to introduce the "indirect" language.

(In *direct* solutions, the pupil is working with what we may call percentage *percepts*. When he takes up the general terms base, amount, etc. of the *indirect* solutions, his working material is what we may call percentage *concepts*. To have *introduced* the subject of percentage by using concept material would have been in violation of elementary principles.)

B. How Developed

1. The Introduction

To teach the pupil to apply the term *base* to whole is a simple matter.

Amount and difference are terms that were acquired in addition and subtraction.

In gain and buying commission diagrams he has "added on" scores of times. The selling price in a gain problem and the "total cost" in a buying commission problem are plainly and easily *amounts*—results from *adding on.*

In loss, selling commission, trade discount, and collecting diagrams he has many and many a time "taken off." The selling price in a loss problem, the net proceeds in a collecting problem, the net proceeds in a selling commission problem, etc. are manifestly *differences.*

Base is new, but it goes easily into the vocabulary.

Amount and difference are perfectly familiar terms whose meanings fit into the descriptions in the diagrams.

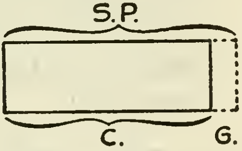
Rate and percentage are so obvious in their reference, in view of the applied percentage experiences that the pupils have had, that the part of the outline which they are used to describe cannot be mistaken. The term rate has, in fact, been in use, and the meaning of percentage is suggested by its etymology.

2. *Diagram Study on Base, Rate, Percentage, Amount, and Difference*

a. WITH DIAGRAMS

These terms are developed through the diagrams.

"You may make a diagram of a gain problem."



The diagrams are made, three or four on the blackboard.

"Show me the whole on the diagram."

"We have another word for whole—base."

"Show me the base."

"There is another word that I want you to have to use—*amount.*"

"What is meant by amount?"

("The answer in addition.")

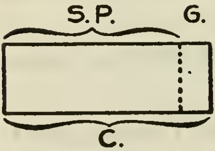
"Yes, it means the result of *adding on.*"

"Show me an amount on the diagram."

The part marked S. P. answers this description and is pointed out. It is developed from this that *amount* in a gain diagram is the selling price.

NOTE I.—The teacher, in the work on the diagrams in the development of the terms base, rate, percentage, etc., will avoid asking the pupils to *show rate* or *percentage* on a diagram. These terms relate to fractional parts. The diagrams show *quantities*. We will say, "Show me the rate;" or, "Show me the percentage," when we are questioning on *outlines*. In this diagram study we confine ourselves to base, amount, and difference.

"Show me the amount."
"What else is this?"
("The selling price.")
"Show me the base."
"What else do we call it?"
("Cost.")
"What does base mean?"
("Whole.")



"You may make a loss diagram."
"Show me the base."
"Why is this the base?"
("It is the whole.")
"Is there an *amount* here?"

It is now developed that there is no amount and that a loss problem could not have an amount because we do not "add on" in a loss problem.

It is recalled that in a loss problem we "take off." The term *difference* is now given.

A loss problem has a difference always.

"Show me the base."

"What does base mean?"
("Whole.")
"What else is this base?"
("Cost.")
"Show me the selling price."
"What else do we call it?"
("Difference.")

The gain diagram and the loss diagram are now reviewed so as to bring out the *base* in each, the amount, and the difference.

"You may make a trade discount diagram."

This is studied in the same way as to base, difference, and amount. The pupil finds that trade discount problems show *differences*, never *amounts*.

The gain, loss, and the trade discount diagrams are now compared as to these terms.

The diagrams of collecting, buying commission, and selling commission, are now taken up in order and discussed in the same way.

b. WITHOUT DIAGRAMS—ORAL EXERCISES

The same discussions are now taken up without diagrams. The aim is to bring out from the pupil's imagery what the *base* is in each class of applied problems (cost, amount collected, or agent's buying price, etc.), which of the classes of problems have *amount*, and which have *difference*.

3. The Terms "Regular" and "General"

It is now necessary to adopt terms that will describe the two forms of outlines and solutions. Pupils should learn to speak of the forms of outlines and solutions that they have been using so far, as "*regular*" form. The base—rate—amount and the base—rate—difference forms of statement may be called "general" forms. In the work that follows, the teacher will speak of a statement as being in *regular* form or in *general* form.

4. Solutions in "General" Form

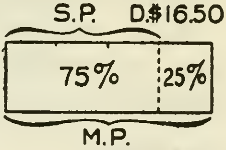
Problems should now be given for pupils to state and solve in the "general" form language. These problems should be in one and then another of the applied percentage subjects. Pupils will have no difficulty in the statements and solutions after the statement of the first problem with *rate* or *percentage* has been worked out.

The diagrams should be required for the first few problems. The diagram is necessary in order that the pupil may get the point that *rate* and *percentage* are *fractions* rather than expressions of *quantity* as are cost, selling price, etc.

NOTE 2.—When the diagram is made, it is discussed as before. The questions are asked (in a gain problem, for illustration), What shows the amount? The child points to it on the diagram. What is its "*regular*" name? ("Selling price.") What shows the cost? The pupil points it out. What is its "*general*" name? ("The base.") What is the gain? This is pointed out on the diagram. What is the *rate?* To answer this question the pupil *does not* refer to the diagram, but to the problem or the outline statement. In his answer he *always* gives the rate of gain. If this rate of gain is not given and is not readily computed mentally, he replies that it is not given. The common fraction or the decimal fraction may be given in answer to the question for rate, the *decimal* should be given in preference to the common fraction if both are given or known. The same applies to the question for percentage. Both rate and percentage relate to *fractions*. The teacher asks, as above, for "*general*" names and for "*regular*" names.

"Marked goods for sale at a certain price but gave a discount of 25 per cent. The discount amounted to $16.50."

"You may make the diagram and the *regular* solution."



1.00, $(\frac{4}{4})$, 1 = M. P. = $66.

.25, $\frac{1}{4}$ = D. = $16.50

.75, $\frac{3}{4}$ = S. P. = $49.50

.25 = $16.50, D.

.01 = $\frac{1}{25}$ of $16.50 = $.66

1.00 = 100 × $.66 = $66, M. P.

75 = 75 × $.66 = $49.50, S. P.

"You may make the *general* solution."

$$\begin{array}{rll}
1.00,\ (\tfrac{4}{4}),\ 1 = \text{B.} & & = \$66. \\
.25,\ \tfrac{1}{4} = \text{R. or P.} & & = \$16.50 \\
\underline{.75,\ \tfrac{3}{4} = \text{Diff.}} & & \underline{= \$49.50} \\
.25 = \$16.50,\ \text{Percentage} & & \\
.01 = \tfrac{1}{25} \text{ of } \$16.50 & & = \$.66 \\
1.00 = 100 \times \$.66 & & = \$66,\ \text{Base} \\
.75 = 75 \times \$.66 & & = \$49.50,\ \text{Diff.}
\end{array}$$

The teacher will observe that the "regular" and the "general" statements differ *only* in the names applied to the terms. The numerical part of the statement and the whole *solution* are alike in the regular and the general forms. It will not be necessary, therefore, to give further illustrations.

After a few problems have been solved in each form the pupil will be ready for seat work without oversight.

Each problem hereafter should specify the form of solution. Double solutions as above will be unnecessary.

SUMMARY.

The aim of this chapter is an outline for the introduction of the general terms base, rate and percentage, amount, and difference to be used instead of the special terms.

In this outline the diagram plan of the regular percentage work is used with the terms base, amount, difference, etc.

CHAPTER V

INTERPRETING THE RESULTS OF SHORT METHOD OPERATIONS

Topic:—How to Know from the Nature of the Operation Whether the Result is a "Whole" or a "Part"

A. Aim

A matter of great difficulty and a source of perplexity and error with students in percentage operations is the naming of certain classes of *results* when solutions are not written out analytically. Let us take, for illustration, the equation .42 =$16.80. We will assume that this is the statement of the discount in a trade discount problem—.42 = D. =$16.80.

The regular analytical solution is—

$$
\begin{aligned}
.42 &= \$16.80\\
.01 &= \tfrac{1}{42} \text{ of } \$16.80 = \$.40\\
1.00 &= 100 \times \$.40 = \$40
\end{aligned}
$$

The *short* process, that which the mature student would use, is the direct division (really partition) of 16.80 by .42 decimally—

$$
\begin{array}{r}
\underline{.42})16.80_{\wedge}(\underline{40} \\
\underline{16\,8} \\
0
\end{array}
$$

The questions at once arise under the short process, What is this 40? Is it the whole, the marked price; or, is it a part, the selling price?

The student should know positively. There should not be the need of testing the matter by computation to see if it will "prove" as the marked price.

The long process (the analytical, above) answers this question. This is one reason for the use of this form of solution exclusively during the development period in percentage, the other reason being the necessity of training in analysis—analysis, written or mental, being the one final test to which nearly every student turns in cases of doubt as to the correctness of his classifications of such results.

This short process must be taught, however, because it is practical and should and will be used. The *mental* "long process" solution with the *written* short process is theoretically the habit that the student should acquire and preserve, but practically this mental analysis will persist with the naturally systematic individual only. The less methodical student and, in fact, all students should be taught to understand the significance of short process operations. This, to enable him to test safely by other than long form analysis a classification of a result.

It is the aim of this step to develop the effects of this and the other (the "of") decimal operation that enter into percentage computations.

Two problems will illustrate the two operations which the pupil must study—

(1) 1.02 = \$14.28. The operation:—

$$\begin{array}{r} \underline{1.02})\$14.28(\underline{\$14}\\ \underline{10\,2}{}^{\wedge}\\ 40\,8\\ 40\,8 \end{array}$$

What is the \$14.28? Is it a whole or is it a part?

What is the \$14? Is it part or is it whole?

(2) .52 of \$65 = The operation:—

$$\begin{array}{r} \$65\\ \underline{.52}\\ 130\\ \underline{325}\\ \$33.80 \end{array}$$

What is the \$65? Is it whole or is it part?

What, in the same way, is \$33.80?

B. The Development

1. Common Fractions

This development is best introduced and the principles illustrated through the use of operations with common fractions.

a. The Equation.

$\frac{3}{4} = 12$. This problem is placed before the pupil for analysis.

The analysis is—(1) $\frac{3}{4} = 12$, part.
(2) $\frac{1}{4} = \frac{1}{3}$ of $12 = 4$, part.
(3) $\frac{4}{4} = 4 \times 4 = 16$, whole.

(1) Special Form in the Solution

For our purposes of development it will be necessary to add a feature to the above analysis. We wish to have the pupil note carefully that in (2) the operation is *division* by the *numerator* of the first member of the equation in (1), and that the operation in (3) is multiplication by the denominator in the first member of (1).

The form, in order to *force* the pupil to think of this while he is working, should be—

(1) $\frac{3}{4}=12$
(2) $\frac{1}{4}=\frac{1}{3}$ of $12=\left(\begin{array}{r}4\\3\overline{)12}\end{array}\right)=4$

(3) $\frac{4}{4}=4\times4=16$

Note 1.—The pupils must discriminate the three statements as (1) the problem, (2) the first analysis statement, (3) the second analysis statement.

"How many steps in the analysis?" ("Two.")

"What is the operation in the 1st analysis step?"

("Division by 3, the numerator of the problem.")

As the pupil answers this question he points to the "$\frac{1}{3}$ of 12" *and* the "$\left(\begin{array}{r}4\\3\overline{)12}\end{array}\right)$."

"What is the operation in the 2d analysis step?"

("Multiplication of the quotient by 4, the denominator of the problem.")

(As the pupil answers this question he points to the multiplier (4×4) in the 2d analysis statement.)

"What is the whole in this problem?"

(2) SPECIAL QUESTIONS

"What is the numerator in this problem?" ("3")

"In which analysis step do we work with the numerator?" ("In the first.")

"What is the work with the numerator in that step?" ("We divide by it.")

"In which analysis step do we work with the denominator?" ("In the second.")

"What is the work with the denominator in that step?" ("We multiply by it.")

b. The "Of" Expressions

$\frac{5}{6}$ of 12.

The analysis is—

(1) $\frac{5}{6}$ of 12, whole.

(2) $\frac{1}{6}=\frac{1}{6}$ of $12=\left(\begin{array}{r}2\\6\overline{)12}\end{array}\right)=2$, part.

(3) $\frac{5}{6}=5\times2=10$, part.

"What is the operation in the 1st analysis step?"

("Division by 6, the denominator of the problem.")

(As the pupil answers he points to the $\left(6\overset{2}{\overline{)12}}\right)$ of the 1st analysis statement.)

"What is the operation in the 2d analysis step?"

("Multiplication of the quotient by 5, the numerator of the problem.")

(As he answers he points to the 5 of the analysis.)

(1) Full Statements

The pupil must now be required to give orally the full statement of what the operations were—

("In this step" (the first analysis) "I divided the 12 by 6, the denominator of the problem. In this step" (the 2d analysis) "I multiplied the quotient by 5, the numerator of the problem.")

"Is 12 a part or the whole in this problem?"

(2) Special Questions.*

"What is the numerator in this problem?"

("5.")

"In which analysis step do we work with the numerator."

("In the first.")

---

*These special questions are of very great importance in this development. *They must not be omitted* in any exercise with common fraction work or in the work with decimal fractions following.

"What is the work with the numerator in that step?"

("We divide by it.")

"What is the denominator?" ("6.")

"In which analysis step do we work with the denominator?"

("In the second.")

"What is the work with the denominator in that step?"

("We multiply by it.")

c. Mixed Exercises.

Many exercises are now given in these two classes of problems, now one or two problems of one of the classes and now one or more problems of the other class. In each exercise the pupil *writes out* the analysis as above and then gives orally the *full statements* of what the operations were, *pointing* as before, as he gives the statement which relates to the first analysis step, to the part or parts of the written analysis where the operation is shown.

(1) Full Statements Before Analysis

After a few exercises of each class the pupils should be required to make these full oral statements of the operations *in advance* of the *written* analysis.

The teacher gives the problem—$\frac{7}{8} = 31.5$.

The pupil, before writing the analysis, states orally what the written analysis will be—

("Seven-eighths is equal to thirty-one and five tenths. One eighth is equal to one seventh of thirty-one and five tenths. I divide by seven, the numerator. Eight eighths is equal to eight times that. I multiply by eight, the denominator.")

It is not necessary in this analysis before the written work to state quotients or products. The essential part is the statement of the operations as above.

(2) QUESTIONS ON PART AND WHOLE

The question, What is the whole? or What is the part? should be asked at the close of each discussion on an analysis. After a few exercises the question should be given in advance of the analysis. When asked in advance, the form should be changed.

If the problem is $\frac{5}{9}=22.5$, the question should be, Is the 22.5 part or whole?

If the problem is $\frac{5}{8}$ of 28, the question should be, Is the 28 part or whole?

Special Questions.—See *a* (2) and *b* (2) above, and footnote on p. 848.

2. *Decimal Fractions*

a. THE ANALYSIS OPERATIONS

The same kind of work is now done with decimal fractions.

(1) .15 = 375.

This problem is placed before the class for analysis.

Before asking for the analysis to be written, the questions should be asked, What is the numerator of this fraction? The denominator?

(These questions should be asked before every analysis because it involves the use of numerator and denominator in exactly the same way as these terms were used in the common fraction work above. Numerator and denominator are important considerations.)

The pupils now write the analysis.

In this analysis the plan of showing in brackets in the first analysis statement the operation that is performed is followed here.

.15 = 375, part.

.01 = $\frac{1}{15}$ of 375 = $\left(\begin{array}{r}25\\15\overline{)375}\end{array}\right)$ = 25, part.

1.00 = 100 × 25 = 2500, whole.

"What is the operation in the first analysis step?"

("Division by 15, the numerator of the fraction.")

(The pupil points as he answers to the $\frac{1}{15}$ of 375 and the $\left(\begin{array}{r}25\\15\overline{)375}\end{array}\right)$.)

"What is the operation in the 2d analysis step?"

("Multiplication of the quotient by 100, the denominator of the fraction.")

(The pupil, in answering, points to the 100×25.)

"Is 375 a part or a whole?"

The "*Special Questions*" under 1 (*a*) and (*b*) above must not be overlooked.

Several similar exercises are given in the same way.

In these exercises, the teacher, instead of asking the questions as to the operations in the two analysis steps should require the *full statements* (1, *b* and *c*) as used in the exercises with common fractions. In fact, this work with decimal fractions repeats the work with common fractions.

(2) .16 of 35. This is given for analysis.

Before beginning to write the analysis, the questions, as above as to what the numerator and what the denominator of the fraction are, are asked.

The analysis is now written—

$.16 = 35$, whole.

$.01 = \frac{1}{100}$ of $35 = \left(100)\frac{.35}{35} \right) = .35$, part.

$.16 = 16 \times .35 = 5.60$, part.

The pupil is required to give the full statements as to the operations in the two analysis steps.

The questions as to whole and part are to be asked.

The suggestions given under the common fraction work above as to asking these questions in advance and changing their forms apply here.

See Questions on Part and Whole under 1 *c* (2) above.

See Special Questions 1, *a* (2) and *b* (2) also footnote p. 848.

The Special Questions and the oral analysis in advance of the written analysis are very important.

(3) Mixed exercises should now be given in each of these two classes of problems.

b. Statements in Advance of Analysis.

The teacher should require the oral statements of the analysis steps in advance of the written analysis.

(These oral statements in advance should be without the results of the operations because the *analysis* is all that is essential.)

.20 = 25.

"Twenty hundredths is equal to twenty-five. One hundredth is equal to one twentieth of twenty five which is *so many*. One hundred hundredths is equal to one hundred times *that*."

(This form of advance analysis is sufficient for all practical purposes. The advantage of this abbreviated form is that it leaves the mind free to concentrate upon the *analysis*.)

c. QUESTIONS IN ADVANCE

The question as to part and whole and as to numerator and denominator should also be asked in advance of the written analysis.

The "Special Questions" must not be omitted. (See footnote p. 848.)

These questions and the analysis in advance are specially important.

3. Parallel Solutions

a. PLAN OF THE WORK

These are solutions in which on one side there is the regular analytical form and opposite this a solution that is also analytical but with these changes—

(1) The analysis uses the decimal fraction.

(2) The pupil writes *only one* analysis statement—the one in which the work is with the *numerator*.

The pupil must state orally in advance of writing the analysis steps—

(1) Whether the quantity (the number) in the problem is part or whole and must mark it.

(2) The analysis steps.

(3) Which analysis step is work with the numerator and which with the denominator.

b. The Form

(P) $.30 = 15$, part.

(N) $.01 = \frac{1}{30}$ of $15 = \left(30\overset{.5}{\overline{)15.\underset{\wedge}{0}}}\right) = .5$, part.

(D) $1.00 = 100 \times .5 = 50$, whole.

(P) $.30 = 15$, part.

(N)* $= \frac{1}{.30}$ of $15 = \left(.30\overset{50}{\overline{)15.00\underset{\wedge}{}}}\right) = 50$

(D) . . . .

The above are the forms of the pupil's parallel solutions. In the first solution he writes the *problem* and marks it (P). He determines next whether 15 is part or whole and writes the word "*part.*"

His next step is to give the *oral* analysis steps (2 *b* above) and to decide which of these is work with the numerator. This being decided he writes the (N) and the (D).

He now writes the two analysis steps opposite the (N) and (D), completing them with "part" and "whole."

The teacher now directs him to make a second solution. The *problem* is written first with its "(P)" and "part," then the (N) and (D) are written ready for the analysis steps.

---

*This space can not be filled until the development work of this section is finished. (See 7 below.) The (P), (N), and (D) in the statements mean respectively the problem, an operation with the numerator, and an operation with the denominator.

The pupil is now told to write only *one* analysis step, the (N) step, but with the *decimal fraction form.* It will be necessary to show him that he is to write only the *solution* part, and that we wish the decimal form as in $\frac{1}{.30}$ of 15 above. In the bracket solution, the pupil must observe that his divisor is a decimal fraction and that he must proceed accordingly.

4. Comparing the Solutions

The pupil's attention is now called to the following facts:

(*a*) That in the 1st solution the operations are with a common fraction and

That we work with both numerator and denominator—*two analysis steps.*

(*b*) That in the 2d solution the operation is with a *decimal* fraction and

That we work with the numerator only—*one* analysis step.

(*c*) That the results in the two solutions are the same.

The form of work for the other class of problems is as shown in the following:

(P) .60 of 45, whole.

(D) $.01 = \frac{1}{100}$ of $45 = \left(100)\frac{.45}{45.\underset{\wedge}{0}0}\right) = .45$, part.

(N) .60 = 60 × .45 = 27, part.

(P) .60 of 45, whole.

(D) . . . .

(N) = .60 × 45 = 27, part.

NOTE 2.—The teacher will observe that, in order to determine whether the 45 in the (P) (or the 15 in the (P) of the problem above) is whole or part and also whether the first analysis step is to be (N) or (D) (the (N) and the (D) being written *before* the written analysis as directed in (*3 b*)), the pupil *must* ask the question as to part and whole and must also give the oral analyses *in advance* of the written analysis work. He would not be able to arrange his (N) and (D) in proper order without stating or thinking the analysis steps. This provides for the advance analysis even in the seat work later when the pupil is not under supervision.

The suggestions under "Comparing the Solutions" above apply here.

5. *Exercises with Mixed Problems*

Three or four exercises with each of the two classes of problems are now given—

$.24 = 30$, $.12$ of 18, $.75 = 27$, $.4$ of 14,
$.45$ of 17, $.8 = 32$, $.40$ of 21, $.25 = 15$.

NOTE 5.—This work, although under the subject of percentage development, should not be confined to work with hundredths. The problems should be in 10ths, 100ths, and, if thought desirable, 1000ths. *Percentage* uses only 100ths, but the truth that this parallel problem work is to bring out is general—it applies broadly. The "Rule" given below is a statement of this general truth.

(P) $.3 = 12$, part.

(N) $.1 = \frac{1}{3}$ of $12 = \left(\begin{array}{r}4\\ 3)\overline{12}\end{array}\right) = 4$, part.

(D) $1.0 = 10 \times 4 = 40$, whole.

(P) $.3 = 12.$

(N) $= \frac{1}{.3}$ of $12 = \left(.3\overset{4\;0}{\overline{)\,12.0_{\wedge}}} \right) = 40$

(D) . . . .

(P) $.002$ of 16, whole.

(D) $.001 = \frac{1}{1000}$ of $16 = \left(1000\overset{.016}{\overline{)\,16.000_{\wedge}}} \right) = .016$, part.

(N) $.002 = 2 \times .016 = .032$, part.

(P) $.002$ of 16, whole.

(D) . . . .

(N) $= .002 \times 16 = .032.$

In each of these exercises the suggestions in "Comparing the Solutions" above must be followed.

After two or three exercises, the pupils should put the facts into full statements to this effect—

"In the first solution, the operations are with numerator and denominator, two statements.

In the second solution, the operation is with a decimal fraction. We work with the numerator only, one statement. The two solutions have the same result."

This statement of the facts should be given in answer to the question, What is the difference between the two solutions? or, Why is the second solution the shorter?

The Explanation.

In connection with the work in which these full statements as above are being brought out, we should have also the explanation of the fact that by operating with the numerator only, in the decimal fraction work, we have the full result as if by operating with the denominator also.

Why in the second solution do we omit the operation with the denominator?

Because, when we place the decimal point after the operation with the numerator, we operate with the denominator.

6. The Decimal Rule

After a few parallel solution exercises for bringing out the full statement and the "explanation" as above, the fact that has now been brought out relative to decimal fraction operations should be embodied in a statement or rule—

RULE.—"In all operations with decimal fractions, we do the work that uses the numerator; the work that uses the denominator is done in placing the decimal point."

The pupil should be required to give this rule very frequently in connection with this part of the percentage work and also later when percentage work is done exclusively with decimal fractions. It should become a part of his *thinking* in fraction operations. The principle involved in it is applied in nearly every decimal computation. No person can successfully use decimal fractions as a tool until this principle has become so familiar to him that it is a part of what he sees in them.

7. Completing the Decimal Solution Outline

Before passing from the "parallel solution" form of work to exclusive decimal fraction solutions the pupil should have experience in completing the decimal solution statement by putting in the fraction which should introduce the (N) analysis step. (See footnote p. 855.)

The pupil has learned through the solution work that the result of the (N) analysis step is the *final result*, it being the same as that of the second analysis step under the *common fraction* solution. He has noted this fact and stated it, first in the full statement following the "Comparing" work and later in the "Decimal Rule." This "comparing," full statement, and "rule" have conveyed the fact that the (N) decimal analysis step *includes* also the (D) analysis step. This makes it necessary for the individual in writing the (N) step to give it for its introductory fraction *the fraction that the second* (final) *analysis step would require if both steps were written.*

Illustrations.

(P) $.9 = \text{S. P.} = \$31.50$, part.

(N) $.1 = \frac{1}{9} \text{ of } \$31.50 = \left(\begin{array}{r} 3.50 \\ 9\overline{)31\underset{\wedge}{.}50} \end{array}\right) = \$3.50$

(D) $1.0 = 10 \times 3.50 = \$35$, whole.

(P) $.9 = \text{S. P.} = \$31.50$

(N) $= \frac{1}{.9} \text{ of } \$31.50 = \left(\begin{array}{r} 35 \\ .9\overline{)31.5\underset{\wedge}{}0} \end{array}\right) = \$35.$

(D) . . . .

(P) .09 of $32, whole.

(D) $.01 = \frac{1}{100} \text{ of } \$32 = \left(\begin{array}{r} .32 \\ 100\overline{)32.\underset{\wedge}{0}0} \end{array}\right) = \$.32$

(N) $.09 = 9 \times \$.32 = \$2.88$, part.

(P) .09 of $32, whole.

(D) . . . .

(N) $= .09 \times \$32 = \$2.88$, part.

In the first problem above the $\begin{array}{r} 3\,5 \\ .9\overline{)31.5\underset{\wedge}{}0} \end{array}$ of the decimal solution is the result as of both (N) and (D) analyses. The (D) analysis, if it had been necessary as a separate statement, would have had as its introductory fraction 1.0. Therefore, the introductory fraction in the (N) analysis should be 1.0, the same as that in the (D) analysis of the common fraction solution above. This would make the (N) statement, when completed, $1.0 = \frac{1}{.9}$ of $\$31.50 = \left(\begin{array}{r} 35. \\ .9\overline{)31.5\underset{\wedge}{}0} \end{array}\right) = \$35.$

The same applies to the second problem. The (N) analysis of the decimal solution includes in itself the (D) analysis. The (N) analysis is the final step. Its introductory fraction should be .09. In all solutions hereafter those in decimal fractions should have this introductory fraction to the (N) analysis.

8. *Exercises in Decimal Solutions*

These exercises are for solutions of problems by the decimal fraction method *without* the parallel common fraction solutions. In connection with this work the "rule" must be required very frequently.

.18 = 44, 33 per cent of 24, 1.2 of 15, .30 = 342, 80 per cent = 92, 102 per cent = 34.68, 105 per cent of 95, 2.25 of 22., etc.

9. *Applied Percentage with Decimal Fractions*

The work should now be in applied work—gain, loss, trade discount, etc.—using decimals exclusively. In this work, the term per cent should be freely used. The *outline* form of statements should be *insisted upon. Every* solution should be given in outline. It saves time in the end, makes it possible to do a greater part of the computation work mentally, and makes for accuracy in results.

As has been suggested before, the pupils should show *all* the parts of a problem in *every* case regardless of problem questions, What was the rate? What was the trade discount? and the like. (See Note p. 823.)

SUMMARY.

In analytical computations the individual necessarily operates with the numerator and the denominator of the fraction separately. This chapter outlines the method by which these operations are combined and the computation shortened without the neglect of the full oral or mental analysis.

The pupil studies the second and third steps in the analytical solutions, noting in each whether the operation is with the numerator or the denominator and noting also the nature of the operation and the result, whether it is a "whole" or a "part."

Computations are now given in which the pupil operates with the numerator alone, retaining and using the decimal point in the operation. He discovers through this work that when the decimal point is retained in the regular numerator operation, the placing of the point in the result serves to effect the regular denominator operation also. He gives orally in connection with this work the full analysis, and is in this way able to interpret the result as a "whole" or a "part."

CHAPTER VI

THE EMPHASIS POINTS IN PERCENTAGE

The pupil should be able to read per cent in all conceivable forms and to translate the common per cent expressions, whether dictated orally or in written form, into common fraction, mixed number, or integral form. 25% of, $33\frac{1}{3}\%$ of, $87\frac{1}{2}\%$ of, $62\frac{1}{2}\%$ of, 20% of, 2% of, 4% of, 5% of, 10% of, $12\frac{1}{2}\%$ of, 40% of, 30% of, 60% of, 50% of, etc. should be instantly interpreted; and, when the partitions indicated are not too difficult, the operation should be instantly performed mentally. There should be as much facility in interpretation and computation when the per cent is written decimally as when it is written as an integer (with "per cent" or %).

He should be able to do the reverse of this mentally. $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{3}{8}$, $\frac{7}{8}$, $\frac{1}{5}$, $\frac{3}{5}$, $\frac{1}{25}$, $\frac{1}{20}$, $\frac{3}{20}$, etc. should be changed instantly to per cent form.

He should be able to solve mentally simple problems in loss and gain, trade discount, collecting, fire insurance, and taxation. He should have much experience in mental solutions of simple problems and in closely estimating the results in problems that are somewhat difficult in each of these subjects. The teacher who would get the best results in percentage teaching must put considerable stress upon mental solutions in each subject after that subject has been thoroughly developed.

The information work in insurance and taxation as outlined, including the definitions of terms in each of the subjects, and the "essential features" in the fire insurance policy as worked out in the schoolroom are very important as lines of practical knowledge.

The power to interpret the result of a single decimal operation as to whether it is a "whole" or a "part" should be one of the enduring results of percentage training. Without it the pupil can never entirely rely upon his own results. This work should have much attention, but no solution should be permitted that is not with *full* oral analysis of the whole problem. This analysis should be the pupil's way of testing the logic of his work. The work of Chapter V, therefore, is of great importance for the development of self-reliance in short processes.

PART VI

BUSINESS FORMS AND INTEREST

CHAPTER I

RECEIPTS, ORDERS, AND NOTES—BUSINESS "FORMS"

A. Receipts

1. Aims

There is no subject in arithmetic to whose development more care should be given than that of these so-called business "forms." A receipt or an order or a note is a simple statement of facts. The more clearly and directly these facts are stated in the document, the better it serves its purpose. The aim in the development, therefore, should be to conduct the work in a manner to impress this fact upon the child by keeping his attention concentrated upon what he wants to say when he is writing a receipt or a promise to pay or an order, and away from any special form of statement—to lead him to be as natural, as free from a set form, so far as possible, as in writing a letter.

2. The Simple Receipt Developed

a. THE INTRODUCTION

This subject should be introduced through an imaginary transaction in the schoolroom—

the purchase of a book or other article by the teacher from one of the pupils.

The book is purchased and paid for in something to represent money. We will assume that the purchase price is $2. It is purchased by Frank, perhaps.

"Now I want each of you to write me a letter as if you were Frank, telling me that you (Frank) have received the money for the book. Such a letter people call a receipt. In a receipt the writer says that he has received something. Receipts are almost always for *money* that has been received. *This* one will be for money."

The response to this will be letters varying in form but somewhat as follows:

"Chicopee, Mass., Jan. 14, 191—

Dear Miss Noland:—

I have received the money for the book.

Yours very truly,

FRANK THOMAS."

NOTE 1.—The pupils have been taught letter writing—the date, the salutation, the complimentary closing, the punctuation, etc. The letters that they write here must be *strictly* in accordance with the plan as to the dating, salutation, closing, punctuation, etc., that is in use in the school. The teacher should insist upon the proper form and punctuation in *these* as well as in the letters written in the regular letter writing exercises.

Some of the letters may be still briefer in statement as, "You paid me for the book," ignoring the suggestion, "In a receipt the writer says that he has received something," in the teacher's remarks when she asked for the letter.

"I am not satisfied with that letter. Some of you did not say how much money you received or the name of the book. I want the whole story. Some of you used the word *received*. That is a proper word to use in a receipt."

These receipts are now laid aside and the pupils try again. After two or three revisions a satisfactory letter will be written—

"Chicopee, Mass., Jan. 14, 191—

Dear Miss Noland:—

I have received from you two dollars. You paid this to me for one copy of Peter Parley's Universal History which I sold to you to-day.

Yours very truly,
FRANK THOMAS."

NOTE 2.—The teacher must keep in mind that the aims in this first step in receipts are—

(1) Full statements of the facts in good English—statements that will identify the transactions.

(2) The use in the letters of the word *received*.

(3) The letter form with proper punctuation.

These letters by the various members of the class will vary somewhat in form. This is as it should be in a composition in which we desire a large degree of freedom of expression. Each is writing a letter telling the same story and each should tell it in his own way. The principal aim is the full story. If the letter tells it in good English and uses the word *received*, as directed, it must be accepted.

b. Owner of the Receipt

The questions should now be brought up, Why is the receipt given? and, Who should keep it?

The teacher should develop that Miss Noland asked for the receipt so that if Frank should forget that she paid him she would have Frank's letter to prove the fact of payment.

The letter should, therefore, be in Miss Noland's possession. She may need it.

The two questions should be asked about every receipt in these exercises—Why given? and, Who should keep it? A part of every exercises in which a receipt or other document is written is to see that it passes into the possession of the person who ought to have it or who owns it.

c. Signatures

The teacher will find in these exercises that many pupils will write the receipts but neglect to sign them or to have the proper person's name signed to them. Very few children realize the importance of the signature. This matter of the need of having the writer's name on a receipt or other paper is something to which the teacher must give special attention and care. The child *must* acquire the signature habit. To develop such a habit, the teacher must not neglect even for *once* to notice whether the child has affixed a signature and the *right* signature to a letter or to one of these receipts.

There must be direct teaching with reference to the effect of a signature. The pupils must be taught that a letter from Charles is not a letter until Charles *signs* it. If James writes a receipt for three dollars, it does not *say* anything, it is not a receipt, until James *signs his name* to it. The question must be asked very frequently, When does a receipt begin to be a receipt? (When the writer signs it.) When does a letter become a letter? (When the writer signs it.) He must be taught, and that strongly, that what a person attempts to say *on paper* is not *said* until the writer *signs* it. All through this work on receipts and in the work on orders and notes which follows, the teacher must give the most careful attention to signatures. It must be impressed upon the pupils strongly and continually that a written statement or order or other document has no force or effect until it has the writer's name affixed. The questions suggested above or similar questions must be in regular use.

d. THE SALUTATION AND THE COMPLIMENTARY CLOSING LEFT OUT

Three or four additional exercises in letter-receipts should now be given in the same way with the same discussion in each case as to what the receipt is for, who should have possession of it, and the effect of the signature.

In these, however, the salutation and the complimentary closing should be omitted. The pupils are told before they begin to write the first one that in real receipts people do not write a salutation or the complimentary closing and that therefore we will omit them in our letters.

A transaction is now suggested between George Field and Charles Bryant, perhaps. These are two pupils in the room. George sells Charles a bicycle for $25 cash.

The writing should be prefaced by questions:

Who received money in this transaction?

Who should give a receipt?

Who should have a receipt?

Who should keep this receipt?

The letter-receipts are now written. Some will be something like the following:

Chicopee, Mass., Apr. 22, 191—

I have received from you twenty-five dollars for a bicycle that I sold you to-day.

GEORGE FIELD.

The teacher now points out that when the salutation is omitted we can not say, "I have received from *you*," because such a receipt doesn't show who "you" is. We must take out the *you* and put in its place the person's name, and in place of the second *you* we must use *him*.

The receipts are now laid aside and new ones are written.

The letters will now have the proper form, somewhat as follows:

Chicopee, Mass., Apr. 22, 191—

I have received from Charles Bryant Twenty Five Dollars ($25)* for a bicycle which I sold to him to-day.

GEORGE FIELD.

Other transactions are now arranged (all between pupils rather than between teacher and pupil) and the receipts written in the same

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*It would be well to teach the pupils to write the numbers and the word *dollars* in all receipts with capitals and to write also the same with numerals—Twenty Five Dollars ($25)—as in the receipt above.

way. The question as to why the receipt is given, who shall keep it, and why it must be signed, must be asked with each.

3. Receipts for Part Payments

a. THE INTRODUCTION

Exercises in receipts should be given in which conditions make the terms "part payment" or "in part payment" necessary. These are transactions in which full payments are not made in money or in which part payments are made at different times as follows:

George buys from Harry a pair of skates for $1.75 giving him $1.25 in money and a knife worth 50¢.

Before writing the receipt the questions are asked—

Who received the money?

Who should write the receipt?

Who should keep the receipt? Why?

The pupils should attempt this receipt without any suggestions being made to them. The incompleteness of a pupil's production makes a suggestion mean more to him than if given before he attempted it.

The receipts will take various forms but none will use the term "part payment" or "in part payment" and some will not tell the *knife* part of the story.

b. The Terms—"Part Payment" and "In Full Payment."

The teacher will call attention to the fact that the $1.25 was a *part* payment only. She will give the pupils the terms that people use—"part payment" or "in part payment"—and direct them to make use of the expression. She should also tell them at the same time that there is another expression, "full payment" or "in full payment," that is used when the money payment is for the full debt. She should tell them also that every receipt hereafter must specify whether the money payment is "in part payment" or "in full payment."

The receipt will now be re-written. The aim, as before, must be a full statement in good English.

The forms of statements in the receipts as finally accepted will differ, but each should cover all of the points in the transaction somewhat as follows:

Chicopee, Mass., March 3, 191—

I have received from George —— One and $\frac{25}{100}$ Dollars ($1.25) in part payment for a pair of skates which I sold to him to-day for $1.75. The remainder, 50¢, he paid by giving me a knife.

HARRY ——

Another problem is now given.

David Parker sells to Gertrude Lindsay a sewing machine for $34. She pays $20 to-day and will pay the remainder to-morrow.

The questions as to who should give the receipt, who should keep it, etc., should be asked.

The receipt when completed will state the facts in some such way as:

Chicopee, Mass., Dec. 3, 191—

I have received from Gertrude Lindsay Twenty Dollars ($20) in part payment for a sewing machine which I sold to her to-day for $34. She will pay the remainder, $14, to-morrow. DAVID PARKER.

After this receipt has been written and given into the possession of the person who should keep it, Gertrude Lindsay, the teacher will announce that it is now to-morrow and that the second payment must be given and the receipt written.

Gertrude Lindsay now pays the $14 and the receipt is written. It must state the following facts:

Chicopee, Mass., Dec. 4, 191—

I have received from Gertrude Lindsay Fourteen Dollars ($14) in part payment for a sewing machine which I sold her yesterday for $34. She paid me the other $20 yesterday.

DAVID PARKER.

In order to bring into use the terms "in part payment" and "in full payment" the exercises must be mixed—some necessitating the use of one of them, others requiring the use of the other.

Raymond Butler sold Guy Taylor a motorcycle for $148. The purchaser paid for it in cash.

Chicopee, Mass., Sept. 15, 191—

I have received from Guy Taylor One Hundred Forty-Eight Dollars ($148). This is in full payment for a motorcycle that I sold him to-day.

RAYMOND BUTLER.

Many other imaginary transactions should be dictated, some of them requiring "full payment" others "part payment." The questions should be asked in each case and the receipt should be placed in the possession, in each case, of the person who should have it.

4. *Receipts for Other Things*

All receipts so far have been for money. The pupil should be made to understand that when one speaks of a receipt, people think (and that properly) that it is a receipt for money. He should not be left, however, with the impression that a receipt is *necessarily* for money. He should learn that receipts may be given for bicycles, for autos, for horses, for articles of clothing, etc.

Exercises should be given in which receipts are asked for for other articles. The following two receipts illustrate a receipt of each kind in one transaction.

John —— sold Henry —— a horse for $140 cash.

Chicopee, Mass., June 1, 191—

I have received from Henry —— One Hundred Forty Dollars ($140) in full payment for my horse, Ned, that I sold him to-day.

JOHN ——

Chicopee, Mass., June 1, 191—

I have received from John —— a horse known by the name of Ned which he sold me to-day for $140. I paid for the horse in cash.

HENRY ——

The teacher should ask frequently for receipts for articles sold and for articles loaned. In the receipts for articles loaned there should be definite statements of the *time* for which loaned and the *purpose*, if given.

B. Orders

Under this head are included orders for goods, orders for service, and checks.

The subject must be taken up in much the same way as that for receipts—by transactions in the schoolroom in order that the child may see their operation.

1. The Oral Order

The introduction must be through an *oral* order in which the teacher directs one of the pupils, whom she selects to be the merchant, to give another pupil some article of merchandise for which she promises to pay later.

a. The Order and Charge

"Robert, you may be a merchant. You have hats, suits, gloves, etc. in your stock. You will please give Charles a pair of your $1.50 gloves and I will pay for them."

Charles is assumed to get the gloves.

The teacher must now take up the matter of what Robert should do. He is a merchant and of course makes a charge on his books when goods are bought and not paid for. The gloves were not paid for.

The teacher should develop that they could not be entered on Charles' account because Charles was not to pay for them. The gloves were *given* to him or he earned them by doing some kind of work for the teacher. They must be entered on the teacher's account, "Jan. 20, one pair of gloves to Charles—$1.50." She promised to pay for them and she should find the charge for them on the account for January which will be sent her by Robert on February 1st.

"To whose account does Robert charge the gloves?"

("To Miss Ward's.")

"Why to Miss Ward's account?"

("Because she ordered them given to Charles and promised to pay for them.")

b. "CHARGE TO MY ACCOUNT"

Several transactions should be given in the same way. In these, however, the order in each case should be given by a pupil and not by the teacher. Each transaction should be discussed in the same way—

To whose account must this be charged?

Why?

Who must pay for this?

To whom must the money be paid?

In these discussions the teacher must use constantly the expression, "Charge to ——'s account." It will take various forms, To whose account must it be charged? Why is it not charged to ——'s account? Why did he charge it to her account? etc. The pupils in making an order (oral) should be encouraged to say, "And charge to my account," rather than, "And I will pay for it."

This brings into the working vocabulary in a proper way the expression quite common in orders, "And charge to my account."

2. *The Written Order*

a. How Introduced

The teacher now asks the pupils to *write* their orders. She suggests a transaction—

"I want Mary Lee to write an order to me asking me to give Lura Carter a copy of Morris's History of Rome. I want each one of you to write the order as if you were Mary Lee."

After the experience in writing receipts there will be few if any members whose orders will not be in proper letter form *with* salutation and complimentary closing somewhat as follows:

Chicopee, Mass., Feb. 10, 191—

Miss Pearl Goodwin.

Dear Madam:—

You will please give to Lura Carter one copy of Morris's History of Rome and charge it to my account.

Yours very truly,

MARY LEE.

The same questions should be asked as suggested in oral orders—

To whose accounts should this book be charged?

Why?

Who must pay for it?

To whom must the money be paid?

The teacher must examine the orders carefully, to see that each has a signature affixed, and the right one.

b. WHAT IS DONE WITH AN ORDER

(*1*) *The Aim*

This subject involves the most important part of the instruction in orders. It is the pupil's large problem.

The aim here is to teach the pupil very carefully what must be done with an order in a transaction—to do this by writing and tracing several of them through their full courses.

To teach this effectively, the first three or four orders, like this Mary Lee order, should be addressed to (drawn upon) the *teacher*. In this way she has a better opportunity to direct the course of the order properly and to call attention to the emphasis points.

(*2*) *Where an Order is Necessary*

Every pupil in the class has been sent to the store to purchase things for cash. Nearly every one of them has had experience also in getting family supplies from the store on credit. Very few of them have had experience in obtaining goods that are to be charged to another family or person. These experiences are the basis for the development of the order.

Before taking up the acting out of the "order" transaction, the teacher should develop, in the manner outlined below, the need of a *written* order when an individual asks for goods on another person's account.

This development should bring out the following three points, the first two relating to facts within the experience of every child in the class, the third to a matter of instruction that the teacher must give with emphasis:

(i) Purchases for Cash.

That, if one of us goes to a store with cash to pay for our purchase, we have no trouble about getting what we want.

(ii) Purchases on Our Family Account.

That many of us, perhaps all of us, have been sent to the store for goods for the family, the goods to be charged to the family account. We had no trouble in getting them because the merchant knew us and knew that some one of the family would pay at the end of the month.

(iii) Relating to Orders.

That if we ask for goods to be charged to another family or person, no merchant will comply without an order—a *written* order.

All pupils will recognize in a general way that this ought to be true, but the teacher must impress it upon them strongly. She must *emphasize* that no goods will be given, under such conditions, unless the person asking for them brings an *order*.

NOTE 1.—The teacher will recognize that this information is to be the pupil's guide to determine who is the *owner* of the order in any case, who must have it. If, for illustration, Lura Carter should try to get the book that Mary Lee ordered given to her, she would find that she could not get it without showing the order. This item of instruction is therefore fundamental in determining the ownership of any given order.

(3) *Tracing the Order*

The teacher now takes up the course of the order through the transaction until the book is paid for.

"Who is to receive the book ordered?"

"Must she pay for it?"

"To whose account is it charged?"

"Who made the order?"

"Will Lura have any trouble in getting the book?"

("She must have the order.")

"To whom, then, must this order be given?"

"Who, then, is the owner of this order?"

"What must she do with it?"

The teacher will now require that the order be given to Lura to whom it belongs. Lura takes it to Miss Goodwin, the teacher, to whom the order is addressed.

Miss Goodwin reads it, notes that it has a signature (that of Mary Lee), and says, "Yes, I will accept this order."

NOTE 2.—The term "accept" must be used. The pupils must be told that "accept this order" as used here means that she will comply with the request and give the book. The term "accept" must be put into constant use.

"If I give you the book you must give me the order. When a person gets goods on an order he must give the order in exchange for the goods. This is the rule. A person cannot get the goods described in an order without giving up the order."

The exchange is now made—the book for the order.

"I must have a receipt for the book."

The teacher now requires Lura to write a receipt, "I have received from Miss Goodwin the copy of Morris's History of Rome.

LURA CARTER."

This receipt is written, not on a paper by itself but *on the order*—on the back of the order across the left end.

Miss Goodwin now states that Feb. 10 is past, that it is now March 1st, and that she wants the money for the book, $1.00.

"Who received the book?"

"Should she pay for it?"

"Who should pay for it?"

("Mary Lee.")

Mary now pays for the book. The teacher must instruct the pupils that Mary when she pays for the book must ask for the order. She must not pay unless she gets it. This is the rule. No one should neglect to demand the order when paying for anything ordered in this way.

This completes the course of the order.

It was given to Lura.

Lura gave it to Miss Goodwin in exchange for the book.

Lura gave a receipt for the book, writing it across the back of the order at the left end.

Miss Goodwin gave it to Mary in exchange for the money given in payment.

c. Language of Orders

The *maker* of the order, Mary Lee, is the person who wrote it.

Mary Lee *drew* the order.

The order was *drawn* by Mary Lee.

Mary Lee *drew* the order "in favor of" Lura Carter.

The order was *drawn on* Pearl Goodwin.

Pearl Goodwin *accepted* the order—agreed to comply with the request.

Lura Carter *indorsed* a receipt on the order—wrote a receipt on the back.

The receipt was *indorsed* on the order—was written on the back of the order.

The teacher should use this language herself in this "order" work and use every means to bring it into regular use by the pupils. It is the common language of business and the pupils should make it their own.

A second transaction should now be suggested and then a third and a fourth. In each of these the order should be drawn by one of the pupils on the teacher. This gives her a better chance to direct the courses of the orders.

The pupils should be told that in orders, as in receipts, the salutation and complimentary closing are omitted.

Each order must be taken through its course very carefully—the presentation by the person in whose favor it is drawn, the acceptance, the exchange of the order for the thing ordered, the receipt indorsed on the order, the payment by the maker for the thing ordered, and the delivery of the order to the maker.

d. QUESTIONS

Questions should be asked to bring into use the language and to test the pupil's knowledge of the course of the order—

Who is the maker?

In whose favor is it drawn?

Who should have the order?

Upon whom is it drawn?

Who must accept it?

Who must indorse the order?

What must he (or she) indorse on it?

("He must indorse the receipt.")

What does indorse mean?

("Indorse means to write on the back of the order.")

Who is the indorser?

("The indorser is the person in whose favor the order is drawn, the person who writes the receipt on the order.")

Who must finally have the order?

("The maker.")

3. Receipting Orders

a. SHORT FORM AND LONG FORM RECEIPTS

Pupils should now be taught that the receipts that they have been indorsing on the order are *full form* receipts. There are also *short form* receipts. A *short form* receipt leaves out all except the signature. In it the person in whose favor the order is drawn simply writes his name on the back of the order. If, for illustration, Lura Carter in the first order had merely written her name on the back of it across the left end, every one would have understood that she meant, "I have received the copy of Morris's History of Rome. Lura Carter." People sometimes indorse in the *long form* and sometimes in the *short form*.

Hereafter, in each exercise the teacher should direct which form of receipt the pupil will use—"Use the long form receipt" or, "Use the short form receipt."

b. ORDERS BETWEEN PUPILS

Transactions in which pupils draw orders on other pupils should now be suggested. In each of these the order is drawn and taken through its full course and the questions asked. In each, the teacher should direct in advance which form of receipt should be used. In each transaction each pupil should be required to write the order in the name of the designated maker—using that name as if it were his own.

The dictations, many of them, should be in form somewhat as follows:

Minnie Chapin draws an order on Louise Snyder in favor of May Harper for a pair of shoes. Write the order and take it through its full course. Short form receipt.

4. Selling an Order

Pupils should be told, after they have had experience with a few orders, that the owner, the holder, of an order may transfer it to some one else.

If, for illustration, James Solin should draw an order on Thomas Pease in favor of Thomas Cronin for a pair of skates, the holder, Thomas Cronin, may sell this order if he desires to Joseph Noland and the latter may take it to Thomas Pease and get the skates.

This transfer is made by indorsing on the back across the left end of the order, "Please give the pair of skates to Joseph Noland. Thomas Cronin." This makes Joseph Noland the owner of the order. He takes it to Thomas Pease, indorses a receipt on the back of the order, "I have received the pair of skates. Joseph Noland." and exchanges the order, thus receipted, for the skates.

Several of the exercises in orders should now direct that the order be sold, the buyer being definitely named each time in the dictation.

C. Checks

1. Orders for Money

Dictations for orders should now be given in which the order is for the payment of money. These orders should be traced through their courses.

Chicopee, Mass., May 8, 191—

Mr. N. P. Carter:

Please pay to Lena Craig Six Dollars ($6) and charge it to my account.

JOSEPHINE HAWES.

Some of the orders should be cashed without transferring. Others should be transferred and then "cashed" by the new owner. Pupils should have much experience in following the courses of transferred orders. The pupil should be instructed to indorse on the order when transferring (on the back across the left end)—

"Please pay to Minnie Fay.

LENA CRAIG."

The pupils should be required to trace the courses of all of these orders.

After three or four of these orders for *money*, the pupils should be instructed to omit the "please" and write the orders—

"Pay to —— —— Seven Dollars ($7) and charge to my account.

——— ———"

The "please" in the indorsement is also omitted.

Some of these orders for money should be drawn on a bank. One or two "bankers" should be selected and given names—"The Charles Flint Banking Co.", "The Mary Morse National Bank."

2. *The Term Check Introduced*

The teacher now introduces the term *check*. An *order* drawn on a bank is a *check*. A *check* is an order drawn on a *bank*. When we draw an order for money on a *bank* we call the order a *check*.

When a check is taken through its course—

The holder takes it to the bank.

He indorses a receipt on the back across the left end—*short form* receipt, the signature only.

He exchanges the check for the money.

The maker has money enough in the bank to pay the check, otherwise the bank will refuse to "accept" it.

The bank deducts the amount of the check from the amount of money that the *maker* has in the bank. In this way the maker of the check pays the bank.

The bank returns the maker's checks, once a month, perhaps, with a statement showing the list of checks paid and how much money is left.

The pupil should be instructed in this to show that the course of a check is *exactly* that of any other order.

3. Or Order, To the Order Of

The pupils have had experience in selling orders that were for goods and have been taught to transfer orders that are for money.

In the orders for goods, the pupil's indorsement in transferring has been, "Please give the ——— to ——— ———," followed by the signature.

In checks, the child was taught to say, "Pay to" instead of "Please pay to"—the "please" being omitted as from the *face* of the check.

The teacher should instruct that checks are very often transferred two or three times. The holder of a check, Mr. X, for illustration, may transfer it to A and A transfer it to B. B takes it to the bank, writes his *short form* receipt (his signature) on the back of the check, and exchanges it for the money.

When Mr. X transfers it to Mr. A, he "orders" the bank to pay the money to Mr. A. He indorses it, "Pay to Mr. A, or order.
X."

The "or order" means, "Pay to Mr. A or to any one to whom Mr. A may order it paid." It gives Mr. A permission to order it paid to some one else—to Mr. B in this case.

Many dictations for checks should be given. In these dictations there should be a transfer of the check to some definite person. These exercises give experience in writing checks to definite persons, in transferring a check to a definite individual with permission to order it paid to some one else (Pay to ——— ———, or order"), and in cashing checks with "short form" receipts.

The pupil should be taught to make equal use of the other "order" form of indorsement. Some people say, "Pay to ——— ———, or order"; others say, "Pay to the order of ——— ———." The meanings are the same. Each gives the new holder of the check permission to "order" the bank to pay it to some other person.

The teacher should require pupils to trace transferred checks through their full courses.

4. Emphasis Points in Orders and Checks

Very much should be made of the "language" of orders in both "order" and "check" work and of the courses:

The *drawer*, the *maker*.

The person "in whose favor," etc.

The person or bank on which it is "*drawn*."

The *holder*, the owner.

Short form receipt on orders.

Full form receipt on orders.

Indorse. (To write on the back of the order or check.)

Indorsing checks "or order."
Indorsing checks "to the order of."
Short form receipts on checks.
Tracing orders through their courses.
Tracing checks through their courses.

Note.—This outline on checks is very elementary but it is as full as should be given to pupils of grammar school age. On the other hand, the information that it gives is information that every child will need in practical life.

D. The Note

1. The Introduction

The development of the note is also through the letter. A note is in reality a letter, without its salutation and complimentary closing, promising to pay some one a certain sum of money.

The subject is introduced by an imaginary loan of money by the teacher to each pupil. This gives to each an opportunity to write a note-letter.

2. "I Promise To Pay"—The Note's Language

The teacher suggests that each pupil wants to borrow five dollars from her for a month.

"Will you pay it back?" ("Yes.")

"When will you pay it?" ("In a month.")

"You *promise* me that you will pay it back in a month?" ("Yes.")

"Well, here is the money. We will play that this is $5 and that I loan each of you that amount."

"I want each of you to *write* me this promise. You must state it in a straight-forward way."

This will bring out letters in which the promises are direct but variously expressed. The teacher examines them. The common form will be about as follows, using that written by Ray Fielding:

Chicopee, Mass., Nov. 5, 191—

Miss Lena Allaire.

My dear Madam:—

I promise to pay you in a month the five dollars that you loaned me.

RAY FIELDING.

"These letters are called notes. They are promises to pay. People speak of written promises to pay as notes."

"Notes, however, like receipts and orders omit the salutation and the complimentary closing."

"I want you to rewrite these notes and be more particular about the time. You should state it by giving the date when you will pay or you should tell me that you will pay me one month from to-day. You may give it in either of these ways. You may now re-write the notes."

The results of the revision will be various but in general in one or the other of the following forms:

Chicopee, Mass., Nov. 5, 1918.

I promise to pay to Miss Lena Allaire the Five Dollars ($5) that she loaned me one month after to-day.* RAY FIELDING.

I promise to pay to Miss Lena Allaire Five

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* If in this work on notes a note written by one of the pupils is poorly arranged, that paper should be dealt with for the English in the same way as the teacher would deal with any other written language exercise.

The arrangement in the insertion of the time phrase "one month after to-day," where pupils have used that expression, should now be given very special attention. One of the notes should be written on the blackboard just as the pupil wrote it (this note, for illustration) and studied as to the proper place to insert this time phrase.

Doubtless the pupils will decide that as written it might modify *loaned*—loaned one month after to-day. They will at least decide that the arrangement is unfortunate. Should we use the phrase as the introduction to our note? In other words shall we say, One month after to-day I promise to pay, etc.? The pupils will decide that this is much less objectionable. Some one may suggest its insertion after the name Allaire, I promise to pay to Miss Lena Allaire one month after to-day, etc. This form of arrangement and that in which the phrase introduces the note may be decided as proper.

Dollars ($5) on Dec. 5, 1918. This is for the money she loaned me.

MONROE UNDERWOOD.

The notes should now be examined as to arrangement—as to their English—and revised (see footnote p. 896) so as to meet one or the other of the forms that have been decided to be acceptable.

3. Interest

The teacher now takes up the matter of interest on the note.

"When I loaned you the money I did not say anything about interest. I intended to let you have it without interest. Every note, however, should have something in it about interest. If the lender is to have six per cent or four per cent or eight per cent or any other rate, or if the note is to be without interest, the note must so state."

The pupils are now asked to rewrite the notes and provide for the statements in it about the interest so that it will conform to the lender's intention as to not making a charge for the use of the money.

This revision will show the statement about interest in different forms. The usual form on the papers will be, There is no interest; or, There is to be no interest. There may be two or three in the short form—Without interest.

Any one of these forms of statement should be accepted.

Another note should now be written. This time each pupil, excepting, of course, the lender, must consider that he or she has borrowed $8 from one of the pupils to be designated by the teacher. This loan is also without interest.

These notes are to be examined in the same way and, when necessary, revised.

One or two more notes should now be written. These *should*, however, be *with interest* at a stated rate. They should be examined and revised as were the others.

NOTE 1.—The fact of the introduction here of this outline for interest must not be taken by the teacher to mean that the subject is to be deferred to the grammar school period. These chapters on interest and business forms must be considered as dealing with separate and practically *independent* teaching units. They must be looked at simply as outlines of methods of instruction in the several subjects and not as in any way suggesting the time or order in which they should be developed.

The subjects as here outlined—receipts, orders, checks, notes, interest, etc.—are not pedagogically consecutive. They can not be ranked as belonging to the work of children of the same degree of mental maturity. Receipts and the introductory interest work, for illustration,—the one properly a part of the work in written English, the other properly an arithmetic subject—

should be assigned to an early intermediate grade. Orders and notes, on the other hand, appeal to more mature minds, to children of the 6th or a later grade.

This work on the development of the note assumes that the pupil understands what interest means and that he has learned how to make simple computations in interest. Chapter II, *A 1-3*, therefore, is an outline to be used in an intermediate grade while Chapter I is an outline that is in part intermediate and in part grammar school in its appeal. This shows how necessary it will be for the teacher to rearrange the units of these outlines, in order that she may assign each to its proper place in the teaching program.

4. After To-Day—After Date

After three or four notes have been discussed, the teacher should take up the word "to-day" in the *time* phrases. It must be taken up in connection with the discussion of one of the notes.

The teacher will introduce the subject by suggesting that she wants to tell them about this word "to-day." In what she says she will bring out as follows:

One month after to-day or six months after to-day is all right, it makes a good note, it is good English, it says just what the writer wants to say.

To-day means the day on which the note is written. If the note is dated November 5, to-day in that note is November 5, everybody so understands it.

Some people say one month or four months or six months after *date*. That means one month or four months, etc., after *this date*. *This date* means the date of the note.

One month after *to-day* and one month after *date* have therefore the same meaning.

People do not say one month after *to-day* although there is no reason why they should not. They use the other form "after *date*."

It would be well for us to say, hereafter, "after *date*" instead of "after *to-day*."

EXERCISES IN "AFTER DATE"

Several exercises should be given in writing notes. Some of these should be with interest at a definite rate and some without interest. In these the pupil should be required to make use of the term after *date*.

5. *Terms*

The following terms should be brought into use:

Maker of the note—the person who promises to pay.

Drawer of the note—the maker.

The note is *drawn* in favor of the person to whom the promise is made.

The *payee* is the person to whom the money is to be paid.

The date of the note.

The date of maturity of a note is the date when the note is to be paid.

The period of maturity of a note is the length of time from the date to the date when due.

The pupils should be given these terms and afterwards on each note they should be questioned—

Who is the maker? Why?

In whose favor is the note drawn?

What is the date?

What is the date of maturity?

What is the period of maturity?

Who is the payee?

The teacher should make the fullest possible use of the terms. In this way they will find their way into the pupils' vocabularies.

6. Selling Notes—Transferring Them

The pupil learned in "Orders and Checks" that such document may be transferred from the owner, the holder, to another person—to a new holder. He learned how to make the transfer by indorsement. The manner of the transfer of a note, therefore, will be understood at once.

The teacher will inform the pupils that a person who has a note that is not yet due may sell it. A note drawn by one of the pupils in favor of another should be taken as an illustration of the mode of transfer—

Chicopee, Mass., Dec. 10, 191—

Two months after date I promise to pay to Mabel Chapin Twelve Dollars ($12) with interest at the rate of 3 per cent.

Lillian White.

The questions relative to date, date of maturity, maker, payee, etc., are asked. The teacher then suggests that Mabel wishes to sell it and that Charles Jennings will buy it. The price agreed upon is $11.75.

Note 2.—There should be no computations in connection with this instruction in note transfers. The proper place for computation work in the sale of notes is in bank discount and present value later. In the exercises here the teacher should arbitrarily set the prices for the sales of the notes. The work here is entirely on manner of transfer.

The pupils will doubtless suggest to transfer by indorsement and that Mabel's indorsement should be, "Pay to Charles Jennings.

Mabel Chapin."

Such an indorsement for transfer is directly in harmony with the method of transferring checks with which the pupils are familiar. They will without suggestion write it across the back of the note at the left end as before.

7. *Or Order—To the Order Of*

The pupils learned the significance of these phrases and made use of them in checks.

a. OR ORDER IN THE NOTES

They should be told that notes nearly always give permission to the holders to sell and transfer them to some one else by writing the promise, "I promise to pay to —— ——, or order"; or, "I promise to pay to the order of —— ——."

They will readily understand that this means "I promise to pay to —— —— or to any person to whom he may order it paid."

b. OR ORDER IN THE INDORSEMENTS

The form of *indorsement*, "Pay to the order of —— ——," which was used in checks should be used with notes also. The pupil learned in his work with checks that this form of expression in the indorsement gives permission for a second transfer.

8. *Writing Trade Notes*

The pupil should now write notes in common life. Mr. Allen sells Mr. James Howard a cow for $80, $30 cash and $50 in two months with interest at 5 per cent.

The pupil is required to write the note. After one or two revisions under the teacher's direction the notes by the pupils will be somewhat as follows:

Chicopee, Mass., March 8, 191—

For value received * I promise to pay to Mr. Allen, or order, two months after date Fifty Dollars ($50) with interest at the rate of 5 per cent. This is in part payment for a cow which I have bought of Mr. Allen for $80. I paid the remainder $30 in cash.

James Howard.

The questions relative to *maker*, *holder*, payee, date of maturity, etc., should be asked on this note.

The note should be sold at a stated price, properly indorsed, paid to the holder, receipted in full, and delivered to the maker.

Another note is written, the transaction being the sale of a lot for $225, a note being given for the full amount and payable in 30 days without interest.

Chicopee, Mass., May 10, 191—

I promise to pay to Leo Leonard, or order, thirty days after date Two Hundred Twenty Five Dollars ($225) without interest, for value received. This is in full payment for a lot that I have bought from Mr. Leonard.

Albert J. Shea.

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*This will not appear in the child's note until after its introduction as outlined in *9* below.

Several exercises should be given in making notes to correspond in this way to the conditions of trades. These notes should be transferred so as to give experience in tracing them through a second holder. They should be paid, receipted, and delivered to the maker.

9. Value Received

In connection with this work of writing notes to satisfy trade agreements and of tracing such notes through their payment by the maker, the teacher should introduce the expression "value received."

The instruction consists in telling the pupil that the expression is very commonly used in notes and that they should use it in theirs. The teacher explains its meaning. It means, For something valuable that I have received.

The question should be asked on every note where the pupil uses it, What is the value received here? In the first of the two notes given in *8* above, the answer to this question should be, The cow. The answer when asked on the second note should be, The lot.

NOTE 3.—The forms of receipts, notes, etc. which have been developed and used in this outline are those which contain *explanatory* matters relating to the transaction of which, in each case, the paper is a record. These explanations are in the nature of, "This is for, etc."

The teacher will bear in mind that this outline is for the elementary school pupil and not for the advanced student. It is a development and as such its scope is necessarily limited. It should not go beyond what should be a minimum requirement in each subject.

As a minimum outline, it should have for its aim habits of thought. Its aim should be to give the pupil the habit, when he is called upon to write a receipt or an order or a note, of going immediately to the exact conditions of the transaction and making his paper a record of those conditions.

The only way in which the habit can be developed of making the conditions of the agreement the first consideration with the pupil when he draws up papers of these kinds is to have *all* papers that are used during the development period prepared with reference to the conditions of definite transactions. Later in his experience he will readily learn to omit now and then the statement of the conditions in a paper and use the conventional form, but the *thought* of the conditions will be there because of the habit.

If our teaching of receipts and notes were based upon the conventional forms, the pupil's thought in writing them would necessarily be centered upon the *form*. If, on the other hand, the development plan forces the pupil to consider in each case the *conditions of the transaction* and to record them as a part of the document, it will become established in his mind that the purpose of any such paper is to tell the story of a transaction. In this way he will acquire the habit of considering at once when he draws up one of these papers how he may make the record accord with that which is to be recorded.

On the other hand, the receipts and notes which have explanatory data, as those of this outline, form a large proportion of the papers of this kind in practical life. The training that the pupil receives through their use, therefore, is that which best meets the needs in life of the individual who has had only elementary school education.

SUMMARY.

RECEIPTS (*A*).
- The development of the simple receipt, (*2*)
- Receipts for part payment (*3*).
- Receipts for things other than money (*4*).

ORDERS (*B*).
- The oral order and the person to whom the goods are charged (*1*).
- The written order, what course it takes, and the order language (*2*).
- Receipting orders (*3*).
- Selling and transferring orders (*4*).

CHECKS (*C*).
- An order for money, check (*1* and *2*).
- "Or order" and "to the order of" (*3*).
- Emphasis points in orders and checks (*4*).

NOTES (*D*).
- I promise to pay (*2*).
- Interest statements (*3*).
- After to-day and "after date" (*4*).
- Note terms and language (*5*).
- Transferring notes and "or order" in the notes and indorsements (*6* and *7*).
- Writing notes in accordance with trade agreements (*8*).
- Value received (*9*).

CHAPTER II

SIMPLE INTEREST

Topics:—Analytical Solutions. Short Form Solutions

A. Analytical Form of Solutions

The word interest is familiar to every child of 9 or 10 years of age through outside associations, and he has at least a vague notion of its meaning.

1. Introduction

The subject should be introduced by a concrete illustration given in the form of a question—What is meant by 5% interest? This the teacher proceeds to answer—It means that if I borrow a dollar from one of you and keep it a year, I must give you back the dollar at the end of the year and must give you $.05 for its use. The next questions are on the meaning of 3%, 4%, 8%, etc., and the pupils are required to make the statements. The exercises now take the form of simple mental problems on the interest on $2, $3, $4, etc., kept by the borrower one year at stated rates; then on $1 for 2 yr., 3 yr., 1½ yr., etc.; and then on $2, $3, etc., for 2 yr., 4 yr., 2½ yr., etc. (See Chapter I, *D*, Note 1.)

a. FEATURES OF THE ORAL WORK

The introductory work should have in view the following essentials of language and imagery:

(1) That the meaning of 6% interest, 4% interest, etc., during the first few weeks of the interest development, must be stated by the pupils in the form of an illustration—If a man borrows a dollar and keeps it one year, he must pay back, etc.; or, If a man lends a dollar and the borrower keeps it a year, the latter must pay back, etc.; or, If I put a dollar in the savings bank and let it remain there a year, the bank must give me back, etc.; and,

(2) That the statement of the meaning of any given rate of interest must, besides including the, "If I borrow," or "If I lend," or "If I put in the savings bank," etc., also include the statement of the return of the borrowed, loaned, or deposited dollar—the "must pay back;" etc. This *paying back* is a very important factor in interest thinking; and,

(3) That the *shortened* meaning, which after a few weeks may be used at times instead of the longer form (—— per cent interest means —— cents for the use of one dollar for one year) must be like the long form, in terms of *cents* for the use of one dollar for one year and never in terms of hundredths.

2. *Written Computations*

The work of written solutions should be introduced after one recitation period devoted to the introduction and the oral solutions in *1*.

Written solutions of interest problems are of two kinds—

1st. Analytical solutions in which the pupils write out the full analysis in every exercise. This form of solution should be used exclusively until the pupil enters the 8th grade; and

2nd. Short form solutions in which the analysis work is preserved in the form of *oral* statements of the significance of each process and combination of processes in the written work. In this way, analysis is not sacrificed in the interests of brevity.

3. *The Forms of the Written Analysis*

Illustrations:

Find the interest on $4 for 3 yr. at 7 per cent.

Int. on $1 for 1 yr. = $.07.
Int. on $1 for 3 yr. = 3 × $.07 = $.21.
Int. on $4 for 3 yr. = 4 × $.21 = $.84.

What is the interest on $8 for 2 yr., 6 mo. at 5%.

Int. on $1 for 1 yr. = $.05.
Int. on $1 for $2\frac{1}{2}$ yr. = $2\frac{1}{2}$ × $.05 = $.12$\frac{1}{2}$.
Int. on $8 for $2\frac{1}{2}$ yr. = 8 × $.12$\frac{1}{2}$ = $1.00.

Find the amount of $30 for 2 yr., 2 mo., 10 d. at 4%.

Int. on \$1 for 1 yr. = \$.04.
Int. on \$30 for 1 yr. = 30 × \$.04 = \$1.20.
Int. on \$30 for 2 yr. = 2 × \$1.20 = \$2.40.
Int. on \$30 for $\frac{7}{36}$ yr. = $\frac{7}{36}$ × \$1.20 = $\underline{\$.23\frac{1}{3}}$.
Total interest for 2 yr., 2 mo., 10 d. = \$2.63$\frac{1}{3}$.
\$30 + 2.63$\frac{1}{3}$ = \$32.63$\frac{1}{3}$, Amount.

4. Distribution of Interest Work

No extended outline for teaching simple interest need be made. It is a subject that should be taken up as early as the fifth grade and carried as a "side line" of work, a "parallel" subject, through that grade and the sixth. It becomes a regular line of work in the 7th grade when the "short method" is introduced.

The work done in the fifth grade should be limited to interest (and amount) for years and easy month periods—2 months, 3 months, 4 months, 6 months, 8 months, 9 months, and 10 months.

The aim of the work in this grade should be the written analysis. It should be introduced *early in the year.*

The form of the pupil's written analysis is shown in *3* above. The teacher should require all solutions to be in analysis form—the arrangement, the full analysis as shown, and the periods after abbreviations being systematically looked after.

The development of the analysis work is very simple and will require but a few days. After this, five or six solutions should be given during each week through the remainder of the year.

In the sixth grade the same analytical form of work should be continued. The work in this grade should be extended to include interest for years with the other month periods (1 month, 5 months, 7 months, etc.) and with easy periods of days—15 days, 10 days, 20 days, 5 days, 25 days, 6 days, 12 days, 18 days, 24 days. One solution each day or six to eight each week will suffice to develop the analysis habit in interest. The aim of this fifth and sixth grade interest work is the development of the power to analyze in interest solutions and to analyze rapidly preparatory to the "short method" work of grades seven and eight.

During the last two elementary school years the pupil is given the short, the cancellation method. With that form of statement and solution, problems involving 1d., 2ds., 7ds., etc., with years and months, may and should have proper attention, but not before the pupil has become proficient in short method work with days that are lower fractions of a month than 12ths.

B. Short Form of Solutions

1. The Year Form

(*a*) The pupil, before completing the elementary school course, must be given much experience in a short method of interest computation. Short form work should be given in the latter part of the 7th grade and throughout the 8th grade.

(*b*) Logically, the short form should differ from the other only in the arrangement of the written work. Otherwise, analysis would lapse and what the long form stands for—the development of power and habit in reasoning out interest work—would be lost.

(*c*) In the long form, the reasoning is written out; in the short form, the *very same reasoning should be required* but in *oral* language.

(*d*) The following illustrates a very short interest method—one that permits the retention of the analysis which the pupil has been accustomed to write out:

Problem.—Int. on $27 for 2 yr., 8 mo., 15 d. at 8 per cent.

(There is, in this form, no written work beyond what the statement and the operations indicated in the statement require. The solutions show (in the quotations) the oral work and (in the parentheses) the various stages of the written statement as the oral work proceeds.)

"The interest on $\$1$ for 1 yr. at 8 per cent is eight cents"—$\underline{(\ \ \$.08)}$. First stage.

"The interest on $\$27$ for 1 yr. is 27 times 8 cents"—$\underline{(27\times\$.08)}$. Second stage.

"2 yr., 8 mo., and 15 d. is equal to $2\frac{17}{24}$ yr."

"The interest on $\$27$ for $2\frac{17}{24}$ yr. is $2\frac{17}{24}$ times that"—$\frac{(65\times27\times\$.08)}{24}$. Completed statement.

The pupil now performs the operations indicated, using cancellation when possible.

The short method should be taught in two forms of statements, the two forms differing in the way the *time* is expressed. After the two forms have been fully developed, the pupil should be given much experience in the solutions of miscellaneous exercises in which he is required to choose for each problem the form of solution that is the more suitable. The form just given may be referred to as the *year* form because the days and months are changed at once to a fraction of a year. The other may be called the *month* form because the years and days are in *every* case reduced to months and then that result to years.

2. *The Month Form*

The month form applied to the problem (the interest on $\$27$ for 2 yr., 8 mo., 15 d. at 8 per cent) would give this form of statement—

$\frac{32.5 \times 27 \times \$.08.}{12}$ The 32.5 is the whole time in months. The $\frac{32.5}{12}$ is this time in years. The oral work which accompanies the preparation of a statement of this kind would be the same as under the *year* form excepting as the form in which the time is expressed would affect it. "The interest on $27 for 1 yr. at 8 per cent is 27 times 8 cents"—$\underline{(27 \times \$.08)}$. "2 years, 8 months, and 15 days is equal to $\frac{32.5}{12}$ years." "The interest on $27 for 2 yr., 8 mo., 15 d. is equal to $\frac{32.5}{12}$ times that"—$\frac{(32.5 \times 27 \times \$.08)}{12}$.

The *month* form is the better of the two in any exercise in which the days would by the *year* method give a difficult fraction.

For illustration:

Problem—Int. on $48 for 1 yr., 1 mo., 1 d. at 2½ per cent.

Statement—$\frac{39.1 \times 48 \times \$.05}{3 \times 12 \times 2}$.

In this statement, it will be observed, the time in months is 13.0⅓. This reduced to an improper fraction is $\frac{39.1}{3}$. Reduced to years it is $\frac{39.1}{3 \times 12}$. In this form of arrangement there could never be a common fraction in either numerator or denominator of the completed statement, thus reducing the chances for error in the computations. An accurate result will depend upon the pupil's power to perform the simple operations of multiplication and division indicated.

In this *month* form the days are invariably reduced to *10ths* of a month. The statement for the problem, Int. on $72 for 1 yr., 5 mo., 19 d. at 3½% would be $\frac{52.9\times72\times.07}{3\times12\times2}$. The time in months would be $17.6\frac{1}{3}$; improper fraction form, $\frac{52.9}{3}$. This time in years is $\frac{52.9}{3\times12}$. The time statement in years in a problem involving interest for 35 days would be $\frac{1.1\frac{2}{3}}{12}$ or $\frac{\frac{3.5}{3}}{12}$ or $\frac{3.5}{3\times12}$. The interest on $84 for 35 d. at 4½ per cent would be the result of the operations indicated in $\frac{3.5\times84\times\$.09}{3\times12\times2}$.

3. Oral Analysis of Statements

When a pupil is asked to analyze or explain an interest statement written on the blackboard, he should do so by pointing to where each oral statement, as he talks, is shown in the written—"This is the interest on $1 for 1 yr." "This" (the indicated product of the interest for one year by the number that represents the principal) "is the interest on— dollars for one year." "This" (all that part of the statement that is used to express the months reduced to a fraction of a year) "is the time." "This" (the complete written statement) "is the interest on —— dollars for —— years."

It is this oral analysis that holds the pupil to habits of reasoning out his interest exercises.

C. Forms of Simple Interest Problems

A good proportion of the exercises in simple interest should be given in note form. Notes for such work should be written, some bearing interest from date, some with interest after a given date or period of time (as, after 30 ds. or after 2 mos. or after a certain date), and some with interest after maturity. Purchases and sales of goods or other property should be suggested with agreements as to interest—interest from date or after a given number of days or months. The pupils should be required to prepare notes that are in agreement with the terms of given purchases and sales and to compute the amount due at maturity on each.

D. Why People Borrow

It is not enough that the child learns to write notes and compute interest on them. He must learn something about the occasion for debts. We teach him that interest is payment for the use of money; but he should know what these uses are, how men can afford to pay interest. Why does this grocer, Mr. A, borrow $1000 and pay the lender $60 a year for its use? How is it necessary or profitable for him to do so? Why does this other man, Mr. B., borrow money to buy or build a home? How is it wise for him to do so? Under what circumstances may another find it to his advantage to pay interest on money borrowed to buy an automobile? Under what conditions of income may a man not unwisely borrow money to buy an automobile for pleasure riding? Under what circumstances of *misfortune* might a man be *compelled* to borrow money and pay interest on it?

These questions should be discussed, of course, impersonally and in a very elementary way, the aim being simply to develop the fact that men borrow either because sickness or other misfortune has made it necessary, or because they feel sure that the use of the money will make for increased business profits. Children of this age (7th or 8th grade) are mature enough to have their attention called to the fact that borrowing money is wise for the borrower when necessary or a source of profit; unwise, when neither necessary or profitable.

SUMMARY.

- Analytical Solutions (*A*):
  - Interest and rates per cent—meaning (*1*).
  - Oral solutions in full analytical language (*1 a*).
  - The forms of written analysis (*3*).
  - The fifth and sixth grades work to be in analytical form (*4*).
- Short Form Solutions—with Cancellation (*B*):
  - The "year" form (*1*).
  - The "month" form (*2*).
  - Oral analysis must be *used* in all short form solutions (*1 c* and *3*).
  - The short form work should be given in seventh and eighth grades (*1 a*).
- Interest problems should be in note form (*C*).
- The ethics of borrowing (*D*).

CHAPTER III

COMPOUND INTEREST

1. Savings Bank Instruction

Savings bank instruction is the proper introduction to the subject because so many of the pupils in our grammar grades are already depositors, have passbooks, and have had interest credits; and because that institution affords about the only illustration of the use of compound interest that most people ever have in practical life.

This instruction involves the form of the savings bank passbooks; how deposits and interest are entered in them; what is meant by "quarter day"; what the effect, in the way of interest, of depositing money on "quarter day" and of depositing on other dates; upon what the interest is computed (what the principal) at the end of the interest period of the local bank (the quarter or the half year according as the interest is added quarterly or semi-annually); and, for the compound interest part, what is meant by compounding interest and how to discriminate between interest that has been compounded and that which is simple. The aim of the instruction, in brief, is to teach the pupil to understand his own passbook, to compute the interest on his deposit, and to know whether any given interest entry is compound or simple.

Class work should consist in ruling paper for use as instruction passbooks; in entering imaginary deposits in these books through one, then a second, and then a third imaginary interest period; in computing the interest at the end of each of the periods and entering these amounts in the books; and in deciding in each case whether the interest has or has not been compounded.

2. *Teaching Points*

The teaching points in the part of the work that has to do with the introduction of the terms compounding, compounded, and compound in order as applied to interest are—

(1) That compound interest is interest on a principal that is in part accumulated interest. There must be interest on interest.

(2) That savings bank interest is *compound* during any "quarter" or other interest period, only when the depositor—

(*a*) Withdraws no money before the close of the period; or,

(*b*) Withdraws less than the amount of the interest credits.

(3) That savings bank interest is regularly simple when the interest credits are regularly withdrawn.

These are not to be taught formally. They are given as suggesting the tests which must be used to determine whether an interest credit on a passbook is simple or compound.

3. Compound and Simple Interest Compared

Compound interest and simple interest computations are alike. In form they are both simple interest. The two subjects differ only in the make-up of the principal. Compound interest, therefore, from the teaching standpoint, is a *term* rather than an *operation.* The aim, therefore, in the class work in savings bank interest is to examine each interest credit to determine whether or not the interest was compounded. When the pupil is able to discriminate, he understands the term and the subject. Interest that has not been compounded is *simple.* Interest that has been compounded is *compound.*

4. Extent of School Work in Compound Interest

The subject should not be considered completed until considerable experience has been given in solving regular simple problems not connected with savings bank passbooks. Some experience should be given in the use of compound interest tables. In all such problems the questions as to whether the interest is compound or simple and why should be asked with regard to the interest for each compound interest period.

SUMMARY.

The instruction to be based on local savings bank practices (*1*).

What compound, compounding, and compound interest mean (*2*).

Compound interest and simple interest compared (*3*).

School work should include:—

Savings bank computations.

Experience with compound interest tables (*4*).

CHAPTER IV

THE PROBLEMS IN INTEREST

Topics:—Finding Rate, Time, and Principal

A. Analyzed from a Teaching Standpoint

1. Their Place in Practical Life

The problems in interest—to find the rate, to find the principal, to find the time—occupy so important a place among the forms of computations which practical life requires that their inclusion in the curriculum of ordinary number training is necessary from the standpoints of both content and form.

Illustrations of Two Common Problems.

(1) Sold a carriage for $200 on three months time without interest. Cash value of this sale at 4 per cent interest?

The comparative value of a time sale of this kind and a cash sale for the same amount is one of our very common business questions. It is one, too, that the child of the grammar school age can understand and should have his attention called to. The subject appeals to him. We speak of it to him, as in the problem above, as a question of, "Which is the better sale and how much, when cash placed in the savings bank would earn 4 per cent? It is one of the "problems in interest"—to find the principal. If we speak of it as Present Worth, we simply call it by its other name.

(2) Sold a lot for $240 at a profit of $20. I had owned the lot 4 years. The rate of profit? The rate of interest on my investment?

In a problem of this kind the rate of profit is by no means an indication of the profitableness of the transaction. In fact, the statement that there was a gain of $8\frac{1}{3}$ per cent is deceptive in the extreme. In the light of interest on the investment the profit disappears. These two questions of rate of profit and rate of interest on money invested are associated so frequently in everyday business, the one to interpret the other, that failure to teach the latter and to associate the two closely would send the child out into life ignorant of a very simple and common application of simple interest as well as of the custom in business of interpreting profits in terms of interest on the investment.

2. *Number of Problems*

The computations in the solution of these problems in interest when described in rule form require four separate and distinct statements as follows:

(1) To find the principal, divide the interest by the interest on $1, etc.;

(2) To find the rate, divide the interest by the interest on the principal at 1 per cent, etc.;

(3) To find the time, divide the interest by the interest on the principal for one year, etc.;

(4) To find the cash value of a future payment whether it is without interest or at a rate lower or higher than that which is obtainable, divide the amount of the debt at maturity by the amount of $1 at the current rate.

3. The Solutions Algebraic—Each Having an Unknown Quantity

These forms of computations can neither be changed nor reduced in number. In the matter of development, however, these four classes of problems present but a single principle of reasoning, a single general form of analysis. Each has what we may call an unknown quantity like that which we represent by x in algebraic equations. In each, the student must assume a quantity, a *one*, to be used in its place. In each, the solution consists in finding the ratio of the assumed to the true quantity—dividing the given interest or amount by the interest or amount which the assumed quantity gives.

4. *"Guess" as a Substitute for the Algebraic "Let"*

The aim of the instruction, therefore, should be to approach the subject in such a way as to lead the pupil to discover the fact that the line of reasoning that applies in finding *rate* applies in the same way in finding *principal* or *time* or *cash value* of a future payment.

Such approach can be effected, as in all development work, only under a proper, simple, introductory language—terms from the child's vocabulary, if possible, that are fully expressive of the mental acts and number processes and so true to number and direct reasoning that, while they may be replaced later by parallel expressions that are more technical, there will be no occasion to repudiate them.

5. *The Need of Such Substitute*

Our usual analysis language in this subject—let the principal be $1 or let us represent the principal by $1; let the rate be 1 per cent or let the rate be represented by 1 per cent; etc.; whatever the forms of expression—is so foreign to anything that has ever been in the child's experience or thought that it is impossible for use in introductory work. The idea that a person may "let" $1 or 1 per cent or 1 year be what it is not, or that a person may "represent" any number or quantity by another number or quantity is something that is as new and strange as the subject itself.

The term that most fully expresses that which we wish the pupil later to read into the technical terms "let" and "represent" is *guess*. I guess that the principal was —— dollars, I guess that the rate was —— per cent, etc., express the thought processes with these problems in interest. Guess is specially valuable also because it goes farther, there being a suggestion in it—almost a challenge—to find out how close to the truth the guess will prove to be. The line of reasoning, the form of analysis involved in "guess" language is the same whether the unknown factor is rate, time, principal, or cash value.

B. Finding the Rate

1. The Introduction

The subject is introduced by a simple problem in finding the rate:

Problem.—The interest on $60 for 2 yr., 6 mo. was $9. Find the rate.

The pupils are left to find this rate each in his own way. They will guess and try and guess and try again until the rate is found. Some will find it in a much shorter time and by fewer computations than others.

After the rate is found, questions must be asked—

What was your first guess? Yours? Yours?

What was your second guess? Yours?

How many guesses did you make?

How do you know that the rate found is correct? ("This rate gives me the right interest—$4.80.")

Another problem is now given.

PROBLEM.—The interest on $48 for 1 yr., 9 mo. was $14. Rate?

The pupils solve in the same way, each trying and trying again until he finds the true rate.

The questions as to first guess, second guess, etc., are now repeated. The question as to how the pupil is to recognize the rate is also asked.

One or two additional problems follow:

PROBLEMS.—Interest on $54 for 2 yr., 4 mo. was $5.67. Rate of interest?

Interest on $75 for 1 yr., 8 mo. was $3.12½. Rate of interest?

The aim in these four introductory problems is (*1*) to introduce our term guess, (*2*) to center the pupil's attention upon the fact that he proves his result by a comparison of his own interest with the interest given in the problem, and (*3*) to make him feel the force of the suggestions for system in guessing that will be given in the next step.

2. *The Way to Guess*

a. THE LINE OF THOUGHT DEVELOPED

The teacher now tells the pupils that they have been guessing in the hardest way, that there is a guess that should always be made no matter if they know that it is not right. We should always, always, always guess *one*.

A problem is now given with the suggestion, Now, I want you to make the right kind of a guess this time.

PROBLEM.—Loaned $80 for 1 yr., 6 mo. The interest was $2.40. Rate?

"What should we guess the rate to have been?" ("1 per cent.")

"You may see if the guess is right."

The computation is made and the interest found to be $1.20.

"Was the guess too high or was it too low?

("It was too low.")

"How do you know that it was too low?"

("The interest should have been $2.40. My interest was only $1.20.")

"How much higher than your guess is the true rate?" ("Twice as high.")

"Prove it to me."

The pupil proves by dividing the $2.40 by the $1.20. The actual process of dividing in long division form is to be required in all cases regardless of the fact, as in this exercise, that the quotient may be found by inspection.

"How did you find how much higher the true rate is?" ("I divided the interest, $2.40, by the interest at my guess rate, 1 per cent.")

"What is the true rate?" ("Two per cent.")

b. The Full Written Work

The problem should now be written on the blackboard in full by one or more pupils.

("I guess that the rate is 1 per cent.

Int. on \$1 for 1 yr. at 1 per cent is \$.01.

Int. on \$80 for 1 yr. $= 80 \times \$.01 = \$.80$.

Int. on \$80 for $1\frac{1}{2}$ yr. $= 1\frac{1}{2} \times \$.80 = \$1.20$.

The true interest was \$2.40.

$\$2.40 \div \$1.20 = 2$. The true rate is 2 times as high.

$\therefore$ The true rate is 2×1 per cent $= 2$ per cent.")

Another exercise is given and then others in the same way with the questions. The final full work in each is written on the blackboard. The pupil's seat written work should be arranged, as he proceeds, in the form as given above. The question, What is the best guess when we wish to find the rate? should be asked again and again in connection and out of connection with exercises like these in finding rates.

3. Rule for Finding Rate

Statement of the Process.

How do we find the rate?

To find the rate we divide the interest by what the interest would be if the rate were one per cent. From the quotient we know the real rate.

Problems.—The interest on $90 for 2 yr., 6 mo. was $11.25. Find the rate.

What was the rate, if the interest on $120 for 2 yr., 3 mo. was $12.15?

Find the rate when the interest on $108 for 1 yr., 9 mo. is $7.01.

C. Finding the Time

1. The Way to Guess Wisely

The problem of finding the time is introduced by writing a simple problem on the blackboard and asking the question, What shall we guess the time to have been?

No suggestion need be made as to the proper guess, the inference from the experiences in finding the rate will be direct. The majority will answer at once that it should be one year.

Problem.—The interest on $84 at 5 per cent is $6.30. Find the time.

"What shall we guess the time to have been? ("One year.")

"Yes, that is the only proper guess."

"You may solve the problem."

2. *The Form of Solution*

The interest on \$1 for 1 yr. at 5% = \$.05.

The interest on \$84 = 84 × \$.05 = \$4.20.

The true interest is \$6.30.

\$6.30 ÷ \$4.20 = 1.5. The true time is 1.5 times as high.

∴ the true time is 1.5 × 1 yr. = 1.5 yr. = 1 yr. 6 mo.

1 yr. = 1 yr.

.5 yr. = .5 of 12 mo. = 6 mo.

Time—1 yr. 6 mo.

3. *Questions*

The questions to be asked in each exercise in this step are—

What is the proper guess as to time?

How did you find how much higher the true time was? (I divided the interest, \$6.30, by the interest for my guess time, 1 yr.)

Problem.—The interest on \$96 at 8% is \$19.52. Time?

("I guess that the time was 1 yr.

Interest on \$1 for 1 yr. at 8% = \$.08.

Interest on \$96 = 96 × \$.08 = \$7.68.

The true interest is \$19.52.

\$19.52 ÷ \$7.68 = $2.5416\frac{2}{3}$. The true time is $2.5416\frac{2}{3}$ times as high.

∴ $2.5416\frac{2}{3}$ × 1 yr. = $2.5416\frac{2}{3}$ yr.

2 yr. = 2 yr.

$.5416\frac{2}{3}$ yr. $= .5416\frac{2}{3} \times 12$ mo. $= 6.5$ mo.
$.5$ mo. $= .5 \times 30$ d. $= 15$ d.
Time—2 yr., 6 mo., 15 d.")

Several additional exercises under the same plan. Simple problems only should be given. We are simply developing the line of thought.

4. *Rule for Time*

Statement of the Process.

How do we find the time?

To find the time we divide the interest by what the interest would be if the time were one year. From the quotient we know the real time.

D. Finding the Principal

1. *The Introduction and the Solution Form*

This problem is introduced after the plan suggested for finding the time. A problem is written on the blackboard, the question as to the proper guess is asked, the pupils solve the problem in form as for the rate and the time, and the question, How did you find how much higher than the guess the true principal was? is asked. (I divided the interest by the interest on the guess principal, one dollar.) Further suggestions as to the solutions and form are unnecessary.

2. *Rule for Principal*

Statement of the Process.

How do we find the principal?

To find the principal we divide the interest by what the interest would be if the principal were one dollar. From the quotient we know the principal.

SUMMARY.

The use of these computations is very common in practical life (*A*, *1*).

The four problems stated—to find the rate, to find the time, to find the principal, to find the cash value (*A*, *3*).

The solutions practically algebraic (*A*, *3*).

The substitution of "guess" for "let," the algebraic term, necessary.

To find the rate (*B*):

- The analysis and the written form (*1* and *2*).
- The rule (*3*).

To find the time (*C*):

- The analysis and the written form (*1–3*).
- The rule (*4*).

To find the principal (*D*):

- The analysis and the written form (*1*).
- The rule (*2*).

CHAPTER V

FINDING THE PRESENT VALUE, THE CASH VALUE, OF FUTURE PAYMENTS—"PRESENT WORTH"

A. The Work Outlined

This, the fourth of the problems in interest, must be prefaced by careful instruction on the subject, almost wholly new to the pupils of this age, of the facts, business practices, and technical terms connected with the discounting of debts due at a future date. It would manifestly be out of place to teach a pupil how to compute the present value of such a debt before he learns that in real life there is such a thing as debts that mature weeks or months after they are incurred, that men frequently agree to settle such debts before maturity, that such settlements are possible usually by discounts without loss or gain to *either party*, that the discount which adjusts such debts thus equitably between debtor and creditor are known as *true* discounts, and that the amount of a debt less this *true* discount is the real value, cash value, present value, or present worth—under whichever name we wish to designate it.

In addition to these matters of debts and business practices, there are the accompanying terms with which the child must become familiar and must learn to apply, such as debtor, creditor, date of maturity, present value, true discount, period of discount, date of discount, etc.

There are, therefore, two separate development lines connected with this subject—

(*a*)* The facts, the business practices, and the language of discounting.

(*b*) How to *find* the cash value—the computation work.

B. The Facts, the Business Practices, and the Language of Discounting

1. Aim

The aim here is to teach how it comes about that there are debts that are due at future dates, how debtor and creditor may settle a debt of this kind before it is due without loss or gain to either, what cash value means, what true discount is, how to find out whether a stated discount is "true" or not "true," and what the various terms—debtor, creditor, cash value, present worth, date of maturity, period of discount, etc.—mean. This information work will require time for digestion and assimilation and, on that account, should run parallel with other lines of work such as simple interest, practice exercises in finding the time, the rate, and the principal, and with other desired reviews.

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* It is not necessary that this work which is almost purely *informational* should be taken up under problems in interest. All the facts and the terms relating to present worth (cash value) might very properly be taken up long beforehand in connection with simple interest. In this way the pupils would be ready for the second topic (How to find the cash value) without delay.

2. *The Development Steps*

PROBLEM (*a*).—A sells B a wagon for $140, the debt to be paid in six months without interest.

a. QUESTIONS TO DISCOVER A FAIR SETTLEMENT BASIS

(These questions are repeated with each of the problems (*a*), (*b*), (*c*), etc.

(1) Was this sale as advantageous to A as a cash sale for the same amount?

(*a*) If the cash had been paid, what could A do with it?

(The final answer must be that he could put the money in the savings bank.)

(*b*) The result to A if he should so deposit the money?

(He would get interest which in 6 mo. would be (at 4%) $2.80. Thus at the end of the 6 mo. he would have in the bank—

Money received from B.....	$140.
Interest from the bank......	2.80
	$142.80

The sale was *not* as good for A as a sale for cash. A cash sale would have been $2.80 better because of this interest from the bank.

A would say *yes* to a suggestion for an immediate settlement for $140 cash. He would be the gainer by such payment.

NOTE I.—The Source of Interest.

The savings bank as the source of interest is by far the best because most pupils have had direct or indirect experience with that institution. They have learned practically that money deposited there earns interest. When we speak of money so placed as bearing interest, our supposition is real to them. To say of money paid to A, He might loan it and get interest, would be a weak illustration because—

(*a*) It would be impersonal—lacking in a definite source of interest income, therefore

(*b*) Pupils would not "feel in their bones" that interest would be a sure thing. There would be lack of vividness.

Distractions to be Encountered

Pupils have had no experiences in this matter of debts that are to be paid in the future except with *store* debts—family accounts. They have learned that these debts are not always easily collected, that merchants are glad sometimes to make concessions on such debts for the sake of settlements. Pupils will bring to the discussion of this subject of cash values this notion that perhaps the creditor may have misgivings as to the payment of the debt at maturity. It will be necessary for the teacher to bar out absolutely all questions or suggestions as to the solvency or honesty or promptness of the debtor. This subject of cash values of debts due at future times has to do with debts that are good—debts that will surely be paid when due. The matter of dealing with doubtful debts belongs, if anywhere, to percentage—to commissions on collecting.

Attempts will be made by pupils to assume that the creditor might, instead of putting the money in the savings bank, use it to pay other debts. This, of course, would not give an *interest* return—the only real *measure* that we have of the advantage or disadvantage of a suggested or actual settlement before maturity. This, too, must be barred out. The teacher may dispose of the suggestion by the statement that this particular creditor is rich and has no unpaid bills. He should not be permitted to be so rich, however, that a little interest from the savings bank would not be acceptable. In these discussions, the assumption must be that neither party would be willing to lose a cent.

(2) Was the transaction as advantageous to B as a *cash* sale for $140?

(*a*) This question *must* also be answered from the standpoint of the savings bank. B has lots of money on deposit there drawing interest. The savings bank interest gives us a *measure* and must be adhered to.

B will have to *draw money from the bank* whenever he makes the payment.

(*b*) The result to B of drawing the money from the bank at once and paying it to A?

The answer to this question must be carefully developed.

He would lose the interest on the money thus drawn out.

This loss of interest would amount to $2.80.

Money paid to A	$140.
Interest lost	2.80
Total cost of the wagon	$142.80

B would say *no* to a suggestion for such a settlement—he would lose money.

(3) Could they agree upon a settlement at once for $138?

(*a*) What would A say?

The reasoning is repeated as in (1)—the pupil computing, as before, the interest that A would receive from the bank. He then formulates the results for A—

("Cash that would be received	$138.
Interest that the bank would pay	2.76
Total from the sale	$140.76

A would say *yes*. There would be a gain to him of $.76.")

(*b*) What would B say?

The reasoning as in (2) would be repeated—the pupil computing the interest that B would lose by drawing the money ($138) from the bank. Then he formulates the result for B—

("Cash for the debt	$138.
Interest that he would lose	2.76
Total cost of wagon	$140.76

B would say *no*. There would be a loss to him of $.76.")

(4) Could they agree upon a settlement at once for $137.25?

(*a*) What would A say?

The reasoning and computation as in (1) and in (3) (*a*), then the statement in form—

("Cash that he would receive	$137.25
Interest from the bank	2.75
Total from the sale	$140.00

A would say *yes*. There would be neither loss nor gain.")

(*b*) What would B say?

The reasoning and computations as in (2) and in (3) (*b*), then the statement—

("Cash to pay the debt	$137.25
Interest he would lose	2.75
	$140.00

B would say *yes*. There would be neither loss nor gain. This being the cash value the teacher questions as follows:

b. Questions on Terms

What is the cash value of this debt? ($137.25).

What is the discount? ($2.75).

What is the total debt? ($140).

What is the interest on the cash value? ($2.75).

Who is the debtor? (B).

Who is the creditor? (A).

What do we mean by debtor?

What do we mean by creditor?

These questions should be used with each of the problems.

Problem (*b*).—George buys from Frank a pair of horses for $420 on 4 months' time without interest. The savings bank pays 4 per cent interest.

The Questions.

(1) Could they settle for $415?

This transaction is discussed in the same way—

(*a*) What would George say?

What George would lose in interest is computed and the statement is written out in full as before. This is followed by a statement of what he would say to such a settlement.

(*b*) What would Frank say?

The result to Frank is computed and the

full statement is written out, followed by a statement of his reply.

(2) Could they settle for $413?

This is discussed in the same way and statements made for George and Frank.

(3) Could they settle for $414.47?*

This is discussed in the same way and the statement for each of the men is made. This is found to be the cash value.

The "questions on the terms" are now asked as before. (See p. 943.)

A question (the 9th) is now added to the list—

What is the *period* of discount? (4 months.)

Problem (*c*).—Sam sells Tom a lot for $282 on 3 months' time without interest. The bank

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*The teacher will understand that, in view of the fact that the problem of finding the present worth or cash value has not been developed and must not be attempted until this preliminary informational work has been finished, no question or suggestion involving it could be permitted. In all of these development discussions the cash value will be *given* to the pupils to prove, as in the problems that have been considered, whether the parties to the trades could or could not settle with loss to neither. This suggested amount in the question, Could they settle for ——? must be computed by the teacher. That there is such a thing even as finding the cash value should not be mentioned.

pays interest at the rate of 6 per cent. The men wish to settle at once.

The Questions.

1. Could they settle for $279?

The regular discussion, the statements, and Sam's and Tom's decisions.

2. Could they settle for $277.26?

The discussion, statements, and decisions.

3. Could they settle for $277.83?

The discussion and the statements show this to be the cash value. The questions on the terms follow—including the 9th, on *period* of discount.

(The teacher should use and teach the pupils to use the term *true discount* in place of the single word discount. Pupils may be told that they will discover later why we have to say *true*.)

PROBLEM (*d*).—On March 1st Tom sells Sam a house for $192 on 8 months' time without interest. On May 1st they wish to settle. The bank pays 5 per cent interest.

The Questions.

1. Could they settle for $188.25? *

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* Most pupils will discuss and state this on the basis of a period of discount of 8 months. The teacher should not interfere until the computations and statements are completed. The suggestion then should be limited to the question, When would Tom put the

2. Could they settle for $187.32?

The discussion and statements show this to be the cash value. The questions on the terms follow.

The 10th, 11th, and 12th questions are now added—

What is the date of the debt?

What is the date of the discount?

What is the date of maturity?

(The question on discount should be, What is the *true* discount?)

PROBLEM (*e*).—On February 10th, A sold B a pair of horses for $372 on 5 months' time without interest. They settled March 10th. The bank pays 6 per cent interest.

This should be taken up in the same way, the pupils being required to try $366.20, $363.72, and $364.71 in order.

The questions should be asked as before, including the 9th, 10th, 11th, and 12th.

c. DEFINITIONS OF TERMS

The development of the subject with the use of the technical terms has proceeded far enough to enable the child intelligently to

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money into the bank? With this question or this question repeated the pupils should be expected to discover the error. Most pupils will discover that the savings bank would have Tom's money from May 1st to Nov. 1st. The corrected discussions and statements will follow.

define the terms and to describe operations. The following definitions are suggested:

What is the cash value of a debt when settled before maturity?

The cash value of a debt is the sum of money which with the interest on it for the period of discount would amount to the total debt at maturity.

What is the true discount on a debt?

The true discount on a debt is the interest on the cash value for the period of discount.

What do we mean by date of maturity of a debt?

The date of maturity of a debt is the date upon which it is due.

What is meant by the date of discount?

The date of discount is the date of settlement of a debt when settled before maturity.

What is meant by period of discount?

The period of discount is the time from the date of settlement to the date of maturity.

If we know the true discount how may we find the cash value of a debt?

We may find the cash value of a debt by subtracting the true discount from the total debt.

Given the cash value and the true discount, how may we find the total debt?

We may find the total debt by adding the true discount to the cash value.

NOTE 2.—These definitions and statements of processes are of very great importance. They should be required hereafter in connection with all discussions of problems involving the real cash value of debts and at other times. They must be called for *very* frequently for many weeks.

Chicopee, Mass., April 15, 191—

PROBLEM (*f*).—For value received I promise to pay to M, or order, seven months after date, One Hundred Forty-four ($144) Dollars without interest. This is in full payment for a horse that I bought from him to-day.

N.

The bank is paying 6% interest.

M. and N. settled Aug. 15th of the same year.

This problem is like the preceding excepting that it is in note form. The same suggestions apply. The pupils should be asked to try as bases for settlement $142.09, $141.59 and $141.87 in order.

The questions and the definitions should follow the discussion and the statements of the cash value.

ADDITIONAL PROBLEMS

A few other problems should be given in which the settlement dates—dates of discount—are (as in (*d*), (*e*) and (*f*)) later than the dates of the debts. Some of these, perhaps *all*, should be given in note form.

NOTE 3.—The notes prepared for these exercises should be, like Problem (*f*), for definite imaginary transactions between individuals in the room. They should be informal, stating the transactions in full. Such informal full statement notes are very common in actual business and are, therefore, valuable here for the impression that they make upon the pupil. Through their use here he learns to think of a note not alone as a promise to pay but as a simple, straightforward, clear, and brief but complete statement of facts.

d. PROBLEMS IN INTEREST-BEARING DEBTS

The development of these problems consists merely in asking the questions as to date of the debt, date of maturity, date of discount, and the total debt *before* the discussion and statements are written. The principal question—the one that may have to be asked two of three times on the first problem—is, What is the *total debt?* This question is sufficient. What will Henry (or Tom or Sam) have to pay when the debt becomes due? is all the suggestion that will be needed to lead the pupils to add the interest to the face of the debt to find the "total debt."

Chicopee, Mass., June 10, 191—

PROBLEM (*g*).—Three months from to-day, for value received, I promise to pay to Henry, or order, One Hundred Sixty-five ($165) Dollars with interest at 3 per cent. This is in part payment for his piano which I bought for two hundred fifty dollars. The remainder I paid in money.

JOHN.

The bank pays 6% interest.

They settled July 10th.

Questions.

Which would Henry rather have, the note or the cash? I want you to *guess* at it.

This question sets the pupils studying the note. They will observe that it draws interest but that the bank would pay a higher rate. The conclusion will be that Henry would be better off with $165 cash than with the note. This precedes the questions suggested above as introductory to work on interest-bearing debts—questions as to date of the debt, date of maturity, date of discount, and the *total debt*.

The total debt is found—$166.24.

The pupils are now asked to try $163.25, $164.81, and $164.59 in order.

The discussions, the statements, the questions (the 12 questions), and the definitions (see (*c*) above) are the same as in the other problems.

PROBLEM (*h*).—Another similar note, the rate of interest at the bank higher than the rate in the note.

Chicopee, Mass., July 20, 191—

Problem (*i*).—For value received I promise to pay to Tom, or order, six months from date, One Hundred Seventy-four ($174) Dollars with interest at 6 per cent. This is part payment for a fence which he built along the north side of my lot on George street. The remainder, sixty dollars, I paid in cash.

Sam.

The men settled up September 20th of the same year.

The bank pays 4%.

This note is treated in all respects like the first interest-bearing note above. The pupils state in advance which they think Tom would rather have, the note or the cash. This will bring to their notice at once the fact that the note draws a higher rate than the bank pays. The pupils will note that it will be worth more than the cash.

The total of the debt at maturity is now found—$179.22.

The pupils try $175.27, $176.90, and $176.86 in order.

The discussions, the statements, the questions on the cash value, etc., when determined, and the definitions.

Problem (*j*).—Another note of the same kind is written and treated in the same manner.

The question of greatest importance now, in view of bank discount that will follow, is, What is *true* discount? True discount is interest on the cash value.

This completes the development of the meaning of cash value. Many exercises should be given in the weeks following, this subject being made a "parallel" with other subjects.

C. How to Find the Cash Value—the Computation Work

In (*B*) the teacher led the child to a knowledge of what cash value means. The computations connected with it, which were merely simple interest exercises, were to aid in the development of the language—the technical terms and expressions of processes.

We are now ready for the *fourth* "Problem in Interest," that of finding the cash value. Before taking up this work, a few minutes should be given to review exercises in finding the time, finding the rate, and finding the principal. (See Chapter IV.)

1. The Development Steps

PROBLEM (*k*).—A sold B two cows for $132 on six months' time without interest. They settled two months later. The bank pays 6% interest.

Each pupil should be required to write the note so that it will conform to the trade agreement, and to affix the proper signature. The date used may be the real date or it may be a date three or four months earlier under the teacher's suggestion.

The pupil should indicate the month and day when the settlement is made, the period of discount, and who should be the holder of the note.

Pupils will begin this solution by guessing at the cash value and then proving the guess. A pupil may guess $129. He will discuss it and make the statement for each man. If it proves not to be the cash value, he will guess again and make proof in the same way.

He should be permitted to make two or three guesses and proofs in this way without hindrance from the teacher. This guessing and proving two or three times puts the individual into the proper frame of mind for the suggestion by the teacher, We learned once that there is a proper way to guess, yet we seem to be guessing in all sorts of ways without plan or rule of any kind.

A suggestion of this kind is all that is needed to recall to their minds that the plan under which they had found the rate, the time, the principal may apply to finding cash value also.

("I'll try cash value one dollar.")

The written statement—

Cash value..................	$1.00
True discount................	.02
Total debt (guess)...........	$1.02
Total debt (real).............	$132.

("My guess was too low.")

$132 ÷ $1.02 = 129.41

("My guess should have been 129.41 times as high.")

∴ 129.41 × $1.00 = $129.41.

The full written statement will now be—

Cash value..................	$1.00
True discount................	.02
Total debt (guess)...........	$1.02
Total debt (real).............	132.

$132 ÷ $1.02 = 129.41.

∴ Cash value = 129.41 × $1.00 = $129.41.

This cash value must now be proved, each pupil finding the true discount and the total debt and making the statements.

Another and then another exercise should follow, pupils being required to arrange the solutions in statement form as shown above. Each cash value (real) should be regularly proved.

2. *Statement of the Process—The Rule*

The pupil should be required after each exercise to state this process.

How do we find the cash value of a debt?

To find the cash value of a debt we divide the total debt by what the total debt would be if the cash value were one dollar. From the quotient we know the real cash value.

The questions, How do we find the rate? How do we find the time? How do we find the principal? and How do we find the cash value? must be asked again and again in connection with the exercises and at *other* times.

D. Emphasis Points in Problems in Interest

The cash value of a debt due at a future time is, as has been stated (Chapter V, *A*) the fourth of the problems in interest. The others are—

Finding the rate.
Finding the time.
Finding the principal.

The emphasis points are the statements of the solution processes and the definitions of the terms answering the questions—

How do we find the rate?
How do we find the time?
How do we find the principal?

How do we find the cash value of a debt due at a future time?

What is the cash value of such a debt?

What is the discount?

What do we mean by date of maturity?

What do we mean by period of discount?

SUMMARY.

- Information that the pupil must acquire before "present value" would have a meaning in his mind (*A*).
- The facts, business practices, and language of discounting (*B*):
  - Finding settlement bases of problems—finding what the two parties, debtor and creditor, in a debt due at a future time could settle for with loss to neither (*2 a*).
  - Discount language (*2 b*).
  - Discount terms defined (*2 c*).
  - Problems in which the debts to be discounted bear interest (*2 d*).
- How to find the cash value of a debt due at a future time—"present worth" (*C*):
  - The computation like that in finding the rate, the time, and the principal (*1*).
  - The rule (*2*).
- Emphasis points in true discount (*D*).

CHAPTER VI

BANK DISCOUNT

Topics:—The Meaning, the Computations, and the Courses of the Notes

A. The Presentation

1. *Information to be Given*

This subject must follow Cash Value (or Present Worth, under whatever name it goes) closely. It may with propriety be introduced after the language development work (Chap. V, *B*) and before that of finding the cash value (Chap. V, *C*) outlined for that subject.

Bank discount needs no development. It should be taken up purely as informational work, the teacher's problem being to inform the pupil—

(1) That men who have notes that are not due, sometimes go to a bank and *sell* them, the bank holding them until they are due.

(2) The bank does not pay what we have been calling "cash value" for the note. It does not use *true discount*. It has another kind of discount which we call *bank discount*.

(3) True discount is interest on the cash value. Bank discount, however, is interest on the *total debt* at the maturity of the note.

(4) Banks pay, not *cash value*, but *proceeds*.

Cash value is found by subtracting the *true* discount from the total debt. Proceeds is found by subtracting the *bank* discount from the total debt.

These explanations require a very few minutes—five minutes, perhaps.

2. *Finding Proceeds*

The teacher then takes up the problems that were used in (*B*) of the Cash Value outline (Chapter V). One of them is written on the blackboard and the pupils are asked the regular questions as to period of discount and total debt. Then they are asked to find the bank discount. Having been told that bank discount is interest on the total debt no suggestion will be required.

The proceeds is then asked for.

Exercises in finding bank discount and proceeds follow.

NOTE.—If the teacher introduces bank discount before the second step (*C*) of Cash Value is taken up she should go no farther with bank discount work than the solution of three or four problems until that step (*C*) has been developed.

3. *Form of Exercises*

These exercises should be in the form of definite imaginary transactions between pupils. These transactions should require notes. In some of them the notes should be for the full amounts involved; in others, for part payments. The notes in some transactions should draw interest; in others they should be without interest.

These notes should be prepared by the pupils—the teacher to see that they are exactly in accordance in each case with the conditions of the transaction and that they have the signature of the maker.

4. Questions

There are certain questions that should be asked upon each note—

(1) Whose property is this note? or,
Who is the owner of this note? or,
Who should be the holder of this note? or,
Who is the maker of the note?
(2) Date of the note?
(3) Date of maturity?
(4) Total debt at maturity?
(5) What was the "value received"?
(6) Meaning of the "or order" (or the "to the order of")?
(7) Who is the payee?

5. Taking Notes Through Their Courses to the Bank

There should be one or more "bankers" in the room to whom holders of notes may sell.

When the "banker" buys a note he must compute the proceeds, the price which he must pay, to know that he pays the proper amount. The seller also should make the computation. The two must agree upon the proceeds. These are the problems for solution.

When the sale to this "banker" is made the holder must indorse it properly by writing across the back of the note on the left hand end—

"Pay to the order of (the banker's name)," and sign his name to this indorsement.* It is simply a transfer of the note to the bank.

The "banker" now becomes the holder of the note—takes possession of it and receives the amount of the debt when it becomes due.

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* The aim in bank discount development is to give the pupil merely the simplest elementary information as to what bank discount is, how proceeds is computed, and how ownership of notes may be transferred. In this matter of indorsements the development introduces but the one form—that in which the holder transfers by indorsing "to the order of." This is the simplest, easiest understood form. It is a form of transfer that he learned in "Orders" and must apply constantly in exercises in which bank checks are used. When the teacher goes beyond this form into indorsements "in blank" or those to bearer or those in non-transferable form, she is in danger of attempting informational work that might better be left to a later period—a period when the pupil takes up regular commercial work.

The matters of dates of notes and dates of settlements are easily managed. The real month of the year may be October. The teacher wishes to have an April transaction. "Let us call to-day April 10th." "Shall we?" ("Yes.") The teacher now writes a transaction for class work on the blackboard with the April 10 date. It provides, perhaps, for a note due in three months with interest. The note is prepared by each pupil, it is properly signed, and the owner (holder) is given possession of one of the notes (selected by the teacher) to be considered as the note. The questions are asked as to owner (holder), maker, date of note, date of maturity, total debt at maturity, value received, and the "or order." The teacher now changes the date—"Let us say that it is now May 10th." "Shall we?" ("Yes.") "You may sell the note to the banker. The rate of interest (discount) we will call 8 per cent." The pupils now compute the discount and the proceeds. The "banker" and the holder must agree upon these amounts. The "banker" makes the purchase, the holder indorses it to him in the proper way, and the "banker" takes the note, paying the former holder the amount of the proceeds. The questions are now asked as to date of note, date of discount, date of maturity, total debt at maturity, period of discount, rate of discount, discount, and proceeds. In this way there may be perfect flexibility in the matter of dates.

6. Parallel Work in Present Worth and Bank Discount

After a few exercises in bank discount, mixed exercises in bank discount and present worth should be given and pupils should be required to make two solutions of each problem. In a transaction given for solution under bank discount, for illustration, the pupil should also find the cash value on the date of the discount. In a transaction in which the cash value is required, the pupil should find, also, what the proceeds would have been on the date of settlement.

Present worth (cash value) and bank discount exercises should in this way be parallel lines of work.

In connection with this work the question should be asked very frequently—What is the difference between bank discount and true discount? The answer is the two definitions —Bank discount is interest on the total debt, True discount is interest on the cash value.

Problems should be given in finding the numerical difference between the bank discount and the true discount on a given note. Any problem prepared for use under present worth may be used under bank discount and vice versa; or, as has been suggested, problems may be for the computation of both of these forms of discounts.

7. Statements of Processes—The Rules

Pupils must learn to state the two processes in bank discount. These must be required again and again for weeks—

How do we find bank discount?

We find bank discount by finding the interest on the total debt at maturity at the bank rate for the period of discount.

How do we find proceeds?

We find proceeds by subtracting the bank discount from the total debt.

B. Emphasis Points in Bank Discount

Statements of Processes

How do we find the bank discount on a note due at a future time?

How do we find the proceeds?

How does the holder of a note make the bank its owner?

To make the bank the owner of a note, the holder indorses and signs across its back an order to the maker to pay the debt to the bank.

Terms Defined

Bank Discount.—Interest at the bank's rate deducted from the total debt on the note at maturity.

Proceeds.—The remainder after deducting the bank discount from the total debt on the note. It is the amount that the bank pays for the note.

To Discount a Note.—To discount a note due at a future time means to *buy* it. The bank discounts it, buys it.

To Indorse a Note.—To indorse a note means to write on the back of the note a receipt, an order, or one's signature.

Date of a Note.—The date when given.

Date of Discount.—The date when the bank buys it.

Period of Discount.—The time from the date of discount to maturity.

Date of Maturity.—The date when due.

Rate of Discount.—The bank's rate of interest. It is the rate at which the bank loans money.

For Value Received.—The expression means, for something of value that I have received.

Or Order, or To the Order Of.—The expression gives permission to the holder of a note to order it paid to a bank or to another person.

SUMMARY.

- The presentation of the subject of bank discount (*A*):
  - Bank discount and true discount compared as to bases of computations (*1*).
  - Exercises for solution to be in note form (*3*).
  - Taking notes through their courses (*5*).
  - Bank discount and "present worth" computations should be parallel (*6*).
  - The rules for the processes (*7*).
- Emphasis points in bank discount (*B*).

CHAPTER VII

AVERAGE

The chapters on business forms would be incomplete without a supplementary chapter on averages. We speak of average profits or losses, average *rates* of profits or losses, average rates of interest, average insurance rates, average amounts of loans, average deposits in saving banks, etc. There are few lines of business in which various forms of averaging do not have an important place.

1. Development Steps

Instruction in average is necessarily divided into three development steps:

(*a*). Computations of averages of quantities that are grouped in single units. One orange at 5¢ and one at 6¢; one horse at $150, another at $200, and another at $210; one box weighing 3 lb., one weighing 4 lb., and another weighing 10 lb. are combinations of single-unit quantities.

(*b*). Computations of averages of quantities that *differ* in the number of units,—some having one, perhaps, and others having two or more. I bought an orange for 5¢ and two for 6¢ each. In this there are one 5-cent unit and two 6-cent units—three units in all in the two quantities. I sold one chicken for $1.35 and three at $1.45 each. In this there is a single-unit quantity and a three-unit quantity—four units in all in the two quantities.

(*c*). The application of average to the computations of costs in various lines of productive activity. An illustration of this may be seen in finding the cost per barrel of my apple crop—

Cultivation and care		$15.00
Pruning		8.00
Fertilizer		10.50
Spraying		19.00
Barrels (175) at 33¢		57.75
Picking and packing		52.50
Thinning the trees		16.00
	Total	$178.75
The crop:	No. 1 Apples	125 bbl.
	No. 2 Apples	50 bbl.
		175 bbl.

175 bbl. cost $178.75
1 bbl. cost 1/175 of $178.75 = $1.02½,
average cost per bbl. (approx.)

2. *Averaging Where the Quantities are in Single Units*

It will not be necessary for the teacher to give any formal instruction in what average means provided she introduces the subject with a simple problem. For illustration, I buy an article for 4¢ and another article for 6¢. What was the average cost? As we are dealing with grammar school children in this subject the average in this problem will be readily given—5¢. Instruction, however, will be necessary in order to develop the form in which the written computation must be made. Pupils must be instructed in writing out in regular form the solution of the problem just given:

4¢ + 6¢ = 10¢

There were two units—two articles.

∴ ½ of 10¢ = 5¢, the average cost.

I bought an article for 6¢ and another article for 7¢. What was the average cost? This solution will be given. Following it the written solution should be made in regular form:

6¢ + 7¢ = 13¢

There were two units—two articles.

∴ ½ of 13¢ = 6½¢, the average cost.

One article cost 6¢, another 7¢, another 10¢. What was the average cost? The written solution of this problem should be:

6¢ + 7¢ + 10¢ = 23¢.

There were three units—three articles.

⅓ of 23¢ = 7⅔¢, the average.

There are 5 marbles in one box, 6 in another, and 9 in another. What is the average number of marbles per box? This problem must be written out also in form:

5 marbles + 6 marbles + 9 marbles = 20 marbles.

There were three units—three boxes.

∴ ⅓ of 20 marbles = 6⅔ marbles, the average number per box.

At this point it will be necessary to take up with the pupils the fact that average usually has to do with quantity that is purely imaginary. In this solution, for illustration, we have 6⅔ marbles. There is no such thing as ⅔ of a marble. A marble separated into parts is no longer a marble. It is worthless for any purpose that a marble would be used for. The 6⅔ marbles, therefore, is something that exists only as an expression. It is of no value to us only for possible computation purposes.

There are 12 boys in this room, 15 in the next room, and 16 in the next room. What is the average number of boys per room? The written solution will be made as before:

12 boys + 15 boys + 16 boys = 43 boys.

There are 3 units—3 groups of boys.

∴ ⅓ of 43 boys = 14⅔ boys, the average number in the rooms.

This is a second illustration to the pupil of the fact that an average is purely an expression of quantity that may not exist by any possibility. There could be no such thing as ⅓ of a boy. It will be necessary to take up several problems of this kind dealing with quantities that could not possibly exist in fractional form. Every child must become accustomed to such expressions as 6⅔ marbles, 14⅓ boys, 25½ girls, 5⅔ horses, etc., in order that he may wear off the effects of the shock and become accustomed to looking at such expressions as purely imaginary.

A wide variety of problems ought to be given in this form of average. The average height of three or four boys, the average weight of several individuals, the average number of cents that several boys may be supposed to have in their pockets, the average width of several boards which differ in width. These problems should involve larger numbers in some cases than those which have been given in the illustrations above. They should also include the averaging of fractional numbers,—for illustration, the averaging of ½, ⅓, and ¾. This would involve the addition of these three fractions and the division of the sum by three, the number of units or groups.

In one box there is ⅚ of a pound of a certain commodity, in another box ⅔ of a pound, and in another ⅞ of a pound. What is the average amount in each box? In the written solution, the pupil should find the sum of these three fractions—57/24. There were three units. ⅓ of 57/24 = 19/24 pounds, the average weight.

It should also include the averaging of compound numbers. One board is 8 feet 3 inches in length; another 9 feet 10 inches in length; the other 12 feet 6 inches in length. What is the average length? The sum of these three numbers is 30 feet 7 inches. ⅓ of 30 feet 7 inches is 10 feet 2⅓ inches.

One box weighs 4 pounds 7 ounces; another, 6 pounds 11 ounces; another, 3 pounds 13 ounces. What is the average weight per box?

4 lb. 7 oz.
6 lb. 11 oz.
3 lb. 13 oz.

13 lb. 31 oz.

14 lb. 15 oz.—the total weight.

There are three units—three boxes.
∴ ⅓ of 14 lb. 15 oz. = 4 lb. 15⅔ oz.—the average weight.

Many exercises should be given in single-unit averages and the problems should cover as wide a range as possible—

The tires of my automobile ran 8243 miles, 7148 miles, 9111 miles, and 8952 miles respectively. The average mileage?

I used during a given period 90 gallons of gasolene. The car ran during that period 1474 miles. What was the average mileage per gallon? (In this problem there are 90 units—90 gallons of gasolene.)

A boy put into the savings bank in January $3.50; in February, $2.75; in March, $4.; in April, $4.25. The average per month?

My 12 hens layed 270 eggs in April, 245 in May, and 255 in June.

The average per hen? (In this problem there are obviously 12 units—12 hens).

The average per month? (In this problem it will be noted that there are 3 units—3 months.)

The average per day? (In this there are 91 units—91 days.)

3. Averaging Where the Quantities Differ in the Number of Units.

(*a*). Teaching Points.

In this class of problems the teacher's work will be to concentrate attention in each case upon (1) the number of quantities, (2) the number of units in each quantity, and (3) the computation "factors." In the computation of the averages the work is like that in 2.

(i). Bought 1 article for 8 cents and 2 articles at 12 cents each.

(*b*). Questions.

How many quantities? (The point to be settled is that there are two quantities here and *three* units.)

How many units? How many units in the first quantity? In the second?

(*c*). Solution Forms.

The statement:

1 × 8 cents =	8 cents.	Cost of the unit.
2 × 12 cents =	24 cents.	Cost of the two units.
Total cost	32 cents.	Cost of the three units.

There are 3 units. $\therefore$ $\frac{1}{3}$ of 32 cents $= 10\frac{2}{3}$ cents, average cost.

(ii). In one box there are 12 lbs. of meal, 2 other boxes have each 10 lb., and in each of three other boxes there are 15 lb. The average?

The questions suggested above bring out that there are three quantities and that in these three quantities there is a total of six units,—1 unit in the first, two in the second, and three in the third.

The statement:

1 × 12 lb. = 12 lb. Wt. of the unit.
2 × 10 lb. = 20 lb. Wt. of the two units.
3 × 15 lb. = 45 lb. Wt. of the three units.

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77 lb. Wt. of the six units.

There are six units.

∴ $\frac{1}{6}$ of 77 lb. = $12\frac{5}{6}$ lb., average wt. per box.

In the same way other computations should be made, the teacher using the questions, as before, to center attention upon the number of quantities, the number of units in each quantity, and the number of units in all.

(iii). Two boards are each 10 ft. long and three others are each 14 ft. long. The average?

(iv). I sold 3 chickens for $1.15 each, 2 for $1.30 each, and 1 for $1.50. The average price?

(v). I had 5 acres of corn that averaged 80 bu. to the acre and 8 acres that averaged 95 bu. The average for the two lots?

(vi). A 2-acre lot produced an average of 230 bu. of potatoes per acre; a 3-acre lot produced an average of 215 bu. per acre. The average yield for the two lots?

(*d*). The "Factors" in a Computation in Average.

The teacher in connection with the work

outlined above should introduce the term "factors" or "the factors" in an "average" problem. *Each problem has two.*

In (i) above the "factors" are the *total cost* and the 3 *articles*. In (ii) the factors are the total weight and the 6 boxes. In (iii) the factors are the total length and the 5 boards. In the problem on p. 972 of the average number of eggs *per hen*, the factors are the total number of *eggs* and the number of *hens* (12). In finding the average number of eggs *per month* in the same problem, the factors are the total number of *eggs* and the number of *months*. In finding the average *per day*, in the same problem, the factors are the total number of *eggs* and the number of *days* (91). This attention to the *factors* is of very great importance for clear reasoning in average. It should not be overlooked in a single solution. The *factors* determine the *character* of the *result*—whether, as in the problem of the eggs, for illustration, it is the average per month or per hen or per day.

(*e*). Proof of an Average.

The proof of the correctness of an average is by solving the problem (finding the total) with the *average:*

I bought one article for 9¢, two others at 10¢ each, and three others at 11¢ each.

In this problem there are three quantities and 6 units.

Statement:

1 × 9¢ = 9¢. Cost of the unit.
2 × 10¢ = 20¢. Cost of the two units.
3 × 11¢ = $\underline{33¢}$. Cost of the three units.
62¢. Total cost.

There are 6 units.

The factors are 6 articles and the cost (62¢).

1/6 of 62¢ = 10 1/3¢, average.

Proof: 6 articles at 10 1/3¢ each = 62¢, total cost. If we had taken the simple average of the prices (9¢, 10¢, and 11¢) our result would have been 10¢. Only the correct average will "prove."

The teacher, besides asking for the number of quantities, the number of units, and the factors, should require frequent proof of the correctness of the average by using it to compute the total.

(*g*). Shortening the Computations.

The following computations in average are given to suggest to the teacher how operations in this subject may sometimes be shortened by the choice of a higher computation unit than 1:

Loaned $5 at 6% and $5 at 8%. The average rate?

The computing unit here may be \$1 or \$5. If \$1 is used there will be two 5-unit quantities.

The statement will be—

$5 \times 6\% = 30\%$
$5 \times 8\% = 40\%$
Total 70%

There are 10 units. The factors are 10 dollars and 70%.

$\therefore$ 1/10 of $70\% = 7\%$, the average rate.

If \$5 is the unit there will be two single-unit (\$5) quantities.

The statement will be.

$1 \times 6\% = 6\%$ (This means a unit sum (\$5) at $6\% = 6\%$)
$1 \times 8\% = 8\%$ (This means a unit sum (\$5) at $8\% = 8\%$).
Total . . 14%

There are two computing units.

$\therefore$ ½ of $14\% = 7\%$, the average rate.

The statement, as is seen, is very much simplified by using a 5-dollar unit, as in the latter, instead of the 1-dollar unit. The choice of the computing unit in no way affects the result.

Loaned \$3 at 4% and \$4 at 5%. The average rate?

In this, there is one possible computation unit, \$1.

Loaned $10 at 6% and $15 at 4%. The average rate?

In this, the unit may be $1 or it may be $5. With the former unit the computation would be 1/25 of 120%. With the latter unit the computation would be ⅕ of 24%.

Loaned $100 at 6% and $150 at 5%. The average rate?

The proper computation unit here would be $50.

The statement would be—2 × 6% = 12%
3 × 5% = 15%
Total 27%

There are five computing units.

∴ ⅕ of 27% = 5⅖%, the average rate of interest.

The average yield of wheat per acre on a 4-acre lot was 30 bushels. The average on another lot of 6 acres was 35 bushels. What was the average per acre on the two lots?

This computation would obviously be shortened by using 2 acres as the unit. The statement would then be:

2 × 30 bu. = 60 bu.
3 × 35 bu. = 105 bu.
Total 165 bu.

∴ ⅕ of 165 bu. = 33 bu., the average yield.

Pupils should have many exercises in computing averages with quantities that may be grouped in larger units. They should find the largest possible units in the quantities and should compute with them.

Questions to be Used.

How many quantities to be averaged here?

What is the computing unit? Could there be a larger unit?

How many computing units in each of the quantities?

How many computing units in all?

(*f*). Exercises for Proof:

A boy bought 10 chickens at 28¢ each and 12 at 32¢ each. What was the average cost?

A man has $800 invested at 5% and $300 at 6%. Find the average rate. What should our unit be?

Two children have each invested in thrift stamps $1.75. Three others have each invested $2.25. What is the average investment per child?

During the months of September and October the average contribution per month to the Belgian Relief Fund by this room was $2.36. During November, December, and January, the monthly average was $3.24. What was the average for the whole period?

The average price of the horses when 2 cost $160 each and 4 cost $190 each?

4 boxes weigh ⅓ lb. each and 3 weigh ¾ lb. each. What is the average weight?

The average yield of my 3-acre lot of potatoes was 220 bushels per acre. The average of my 4-acre lot was 275 bushels. What was the average for the two lots?

4. *The Application of Average to the Computations of Costs.*

(*a*). Aim.

A small but very important item in the practical training of every child in number, is instruction in the part that the computation of averages has in the activities of every day life. He should know the use that is made of it as a factor in practical business in determining the unit costs of industrial products.

No child; until his attention is called to it in carefully prepared lessons, ever thinks of the fact that the manufacturer of the shoes that he wears on his feet must have some practical means of knowing definitely what the cost of these shoes is before he can, with safety, determine the price at which he can afford to sell them. He should learn that the producer of any product of the industrial activity—the dairyman, the stockman, the fruit grower, the grain raiser, the manufacturer—must know beyond question in the interests of safe business the full average cost, including every form of expenditure up to delivery in the market, of the article or commodity or structure which he produces.

The child should recognize that the manufacturer of plows must know the total expenditure for lumber, steel, paint, labor, cartage, heat, power, office help, salaries, and other items in the manufacture of the two hundred or three hundred or four hundred plows that have been made within the computation period. He should recognize that this total cost, divided by the number of plows manufactured within the period is the average cost of a single plow, and that from this average the manufacturer knows, definitely and positively, at what price the plows must be sold to realize a profit.

The child's attention must be centered upon the fact that the contractor who built the cement walk along the front of the school building knew positively when he submitted his bid for the construction of this walk what the total average cost of a square yard of such a walk would be. He had built such walks before. He had computed the cost per square yard in each. Repeated experiences had verified the accuracy of his computations. He made his bid on the basis of these averages.

The aim of this work is not to teach the child to recognize all the items that enter into the production of any particular article or commodity but to know the *general plan* that is followed in the computation of average costs. His knowledge of the items of expenditure in the manufacture of a pair of shoes, for illustration, may be entirely superficial, but such knowledge of cost items in the article is entirely sufficient for instruction purposes here. It is enough if he notes the items that are *apparent* from the examination of the shoe, and recognizes, when the suggestion comes to him, that the items of leather and cloth and eyelets and nails and thread are after all not the only items of cost, but that power and heat and light and cartage and labor and boxing and packing and shipping and office help and office supplies and salaries of officers are rightfully items which must *also* be taken account of in finding the actual expense that the pair of shoes represents.

Nearly every intelligent child of 11 or 12 years of age will recognize, when heat, light, repairs, depreciation in tools, salaries, etc. are suggested as items of cost in the manufacture of the shoes or the chair or the sled or the plow or the gun or the skates or the tennis racket, that there are properly these items of expenditures and that in reason they must be regarded as items in the cost of the article. Our aim in the introductory problem is to suggest one or two of these items and then wait to see if there are not pupils who will, under the impetus of that suggestion, anticipate some of the others. It will be found that the pupils *will* anticipate some of these indirect items after one or two of them have been suggested.

(*b*). Work, How Conducted.

The instruction must be conducted in the form of class exercises and should consist almost entirely in listing the items of expenditures in one and then another and then another class of articles under the teacher's guidance. The child has had, in the two classes of computation problems in average that have been developed, enough experience to make assistance in computing these averages unnecessary.

One or two exercises should be given at first in which the amount of *each item* is stated, the totals of the items computed, and the average found. The following will illustrate:

The Concrete Walk.

The average cost per square yard.

24 bbl. Asphaltum at $9.80	$235.20
32 tons Crushed Rock at $1.28	40.96
12 loads Sand at $2.00	24.00
Fuel	10.00
Labor	60.60
Overseer	20.00
Kerosene	.60
Teaming	10.00
Clerical work and supplies	4.00
Depreciation of tools	4.00
	$409.36

Number of square yards of walk—615.

Statement:

615 sq. yd. cost $409.36

1 sq. yd. cost 1/615 of $409.36 = $.66 (approx.) average.

(*c*). Forms of Statements.

(1) My Apple Orchard.

The average cost of my crop per barrel.

Cultivation and care	$ 30.00
Pruning	14.00
Fertilizers	19.50
Spraying	28.00
Barrels for the crop (410 at 30¢)	123.00
Picking and packing	115.00
Marketing	45.40
Total	$374.90

The crop: No. 1 Apples...300 bbl.
No. 2 Apples...110 bbl.
Total........410 bbl.

Statement:

410 bbl. cost $374.90

1 bbl. cost 1/410 of $374.90 = $.91 18/41, average.

(2). The Cement Walk.

The average cost per square yard,

Cement......................	
Crushed rock.................	
Sand.......................	
Coloring material............	
Excavating..................	
Filling.....................	
Labor......................	
Trucking...................	
Clerical work................	
Wear and tear of lumber and tools	
Total............	$460.25

Square yards of walk. . . 201

Statement:

201 sq. yd. cost $460.25

1 sq. yd. cost 1/201 of $460.25 = $ average.

This is the form in which the problems should be worked out. The pupils should suggest, under the teacher's guidance, the items that appear to enter into the construction. The teacher should suggest a total and a number of yards of walk that will give a reasonable average cost per square yard.

Note.—The averages in the exercises may be far from actual costs in the articles which are under computation, but care should be taken that the costs are not impossibly high or low. This total cost and number of yards in the cement walk problem are based on a bid in actual construction work. It allows for a good profit to the contractor.

The aim of the work in an "average cost" problem is fully realized when the child under guidance has made a rough outline of items, as in the walk problem, and computed the average—1/201 of $460.25.

(3). My Herd of Cows.

The average cost of feed and care per cow per year.

Hay to feed them............

Grain......................

Ensilage...................

Pasturage..................

Salt.......................

Green fodder...............

Expense for care............

Depreciation in tools and machinery..................

Repairs on stable............

Fire insurance...............

Total cost....... $3042.50

Number of cows . . 35.

Statement:

35 cows cost $3042.50

1 cow cost 1/35 of $3042.50 = $, average.

(4). The Plow.

Lumber used in the plows....	
Iron..........................	
Material in making the castings and in machining.....	
Paint..........................	
Labor..........................	
Wear and tear on tools and machinery	
Coal for heating and power....	
Repairs on buildings..........	
Upkeep of teams for cartage...	
Insurance.....................	
Office help and supplies.......	
Salaries of officers............	
Total..............	$3879.25

Number of plows produced.......162

Statement:

162 plows cost $3879.25

1 plow cost 1/162 of $3879.25 = $, average.

(5). The Potato Crop.

Rental value of the land.......	
Plowing......................	

Harrowing. . . . .
Seed at $8.00 per barrel. . . . .
Fertilizer. . . . .
Planting . . . .
Cultivating. . . . .
Spraying. . . . .
Digging. . . . .
Hauling to market . . . .
Total. $138.90

Number of bushels of potatoes. . 265.
Statement:

265 bu. cost $138.90
1 bu. cost 1/265 of $138.90 = $, average.

(6). The Pair of Shoes.

The average cost per pair.
Leather for the lower part. . . .
Leather for the upper part . . .
Leather for the soles. . . . .
Cloth for lining . . . .
Eyelets . . . .
Thread . . . .
Nails. . . . .
Material for finishing and polishing . . . .
Power. . . . .
Heat. . . . .
Light. . . . .

Cartage....................	
Labor......................	
Depreciation in tools and machinery...............	
Boxing and packing material.	
Shipping...................	
Office expenses...............	
Insurance..................	
Advertising................	
Salaries of officials...........	
Other items................	
Total............	$14428.50

Pairs of shoes produced..6120

6120 pr. of shoes cost $14428.50

1 pr. cost 1/6120 of $14428.50 = $, average.

5. Sources of Problems.

The activities of every community offer suggestions for exercises in cost averages. The farmer, the poultryman, the gardener, the dairyman, the stockman, the manufacturer—all have their problems of the costs of products. The teacher needs only to look about her to find abundant material for her class work—material for average costs of products, and material for the other applications of average. The most valuable forms of exercises are those that have to do with the interests of the people of the community which the school serves.

SUMMARY

1. The three teaching steps outlined.
2. Averaging single-unit quantities.
3. Averaging quantities that differ in the number of units.
4. Average applied to cost computations in productive activities.

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